An Investigation into the Moment of Inertia of a Hollow, Cylindrical Rod

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Abstract

This paper presents an experimental determination of the moment of inertia of a hollow cylindrical rod using a bifilar pendulum. The bifilar pendulum is a torsional oscillation system in which a rigid body is suspended by two parallel strings, allowing rotational inertia to be extracted from the dynamics of its oscillation. The period of torsional oscillation T was measured across eight values of string separation r $(0.10m\ to\ 0.45m)$, with all other parameters held constant. The theoretical relationship $T = \frac{2\pi}{r} \times \sqrt{\frac{IL}{mg}}$ predicts an inverse proportionality between T and r, confirmed experimentally by a power-law fit with $R^2 = 0.9952$. Linearisation of T versus $\frac{1}{r}$ yielded a best-fit gradient of $0.6206\ m \cdot s$ $(R^2 = 0.9988)$, from which the moment of inertia was derived as $I = (2.01 \pm 0.063) \times 10^{-2}\ kg \cdot m^2$. The bifilar pendulum method determines rotational inertia without requiring knowledge of the object's internal geometry, making it applicable to objects of complex or unknown shape. This is particularly advantageous for hollow or geometrically irregular objects where direct calculation of rotational inertia from physical dimensions alone is impractical. Sources of systematic error including elliptical motion, manual timing, and string friction are evaluated quantitatively. A theoretical comparison reveals a discrepancy between the experimental and predicted values, attributed primarily to non-purely-torsional motion during oscillation, which is discussed in the context of the method's limitations and directions for future refinement.

1.1 Introduction

The bifilar pendulum system is an intricate construct of oscillatory dynamics, serving as a bridge between theoretical underpinnings and practical implementations. It can be attributed to engineering domains (where the system provides a stability analysis of mechanical vehicles) and material science (where the pendulum's oscillatory motion allows density assessment of different materials). This system has also gained importance in measuring the moment of inertia of a test object, extending to aircrafts, specifically small, unmanned air vehicles.

The period, or time taken for one complete oscillation of the bifilar pendulum, is an intriguing aspect of exploration because of a multitude of influencing factors addressed later. This stimulates an intellectual intrigue propelling an investigation of the interplay between the distance of the two strings suspending a mass and the mass's oscillation period. This paper is worthy of investigation as it paves an inquiry that seeks to decipher the complex dynamics embedded within the bifilar pendulum's behavior.

This paper seeks to examine how the relationship of the distance between the strings suspending a bifilar pendulum and its period of oscillation be used to determine the moment of inertia of a hollow, cylindrical rod, utilizing the scientific method as a foundation for this investigation.

1.2 Background Information

1.2.1 Oscillatory Motion and Period of Oscillation

Oscillatory motion is the phenomenon whereby a mass undergoes a to-and-fro motion from an equilibrium position. The time to complete one cycle of oscillation is defined as its "period".

1.2.2 Bifilar Pendulum

The bifilar pendulum is governed by principles of oscillatory motion, outlying Simple Harmonic Motion, equilibrium dynamics, and torque, gravitational and tension forces.

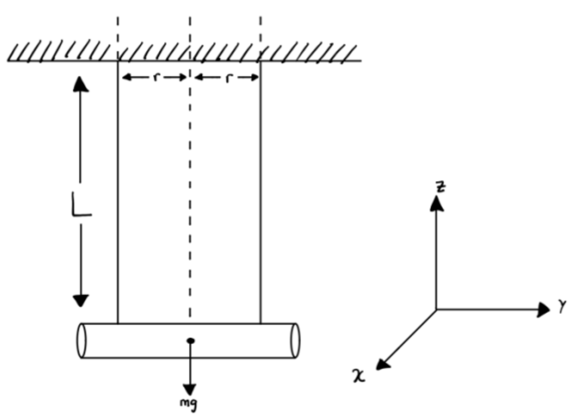


Figure 1: Bifilar Pendulum in Equilibrium [Student's Own Work]

Figure 1 depicts a bifilar pendulum system in rest, suspending a cylindrical rod of mass m kilograms, having its center of mass in its geometrical center (Circus of Physics 4:03). The apparatus hangs the rod a length L meters away from the pivot, with two strings being separated a distance $2r$ meters away from one another and r meters each from the relative location of center of mass.

This system involves motion in the *x-y* plane. Rotational dynamics and oscillations around the fixed pivot point adds a *z* plane in three-dimensional space.

1.2.3 Moment of Inertia (MoI) for Solid Rods

The moment of inertia for an object is defined as its ability to resist angular acceleration (Admin *Moment of inertia*); this resistance is proportional to how much torque force is required for angular acceleration in a rotational axis.

For a solid rod, the moment of inertia about the center can be calculated by using the formula:

$$I = \frac{1}{12}ml^2$$ [1]

Where:

- I is the moment of inertia (kgm^2)
- m is the mass of the rod (kg)
- l is the length of the rod (m)

A solid rod refers to a thin, cylindrical structure that does not have any holes, voids, or cavities. Solid rods are used to study concepts of rotational motion, moment of inertia, and oscillations because of its uniform mass distribution. Calculations are simpler and values such as the moment of inertia are constant as compared to objects with more complex geometrical properties or non-uniform mass distribution.

For a hollow cylindrical rod rotating about an axis perpendicular to its length through its centre of mass, the full formula is:

$$I = m\left[\frac{1}{4}(r_1^2 + r_2^2) + \frac{1}{12}l^2\right]$$

r_1 is the outer radius and r_1 is the inner radius. When $l \gg r$ (a long, thin rod), the radial terms are negligible, and this reduces to the solid rod approximation above. In this investigation $l = 0.655m$ and $r_1 \approx 0.035 - 0.040\ m$, so $l^2 \approx 0.429\ m^2 \gg R^2 \approx 0.0014\ m^2$. The bifilar method is therefore particularly valuable for hollow objects: it determines I directly from oscillation dynamics without requiring knowledge of the internal geometry.

1.2.4 Small Angle Approximation

Small-angle approximation is the process in which trigonometric ratios can be simplified when an angle is small (less than or equal to 15 degrees/0.26 radians).[2] Small angle

[1] “Mass Moment of Inertia.” *Engineering ToolBox*, www.engineeringtoolbox.com/moment-inertia-torque-d_913.html. Accessed 28 Aug. 2023.

[2] Urone, Paul Peter, and Roger Hinrichs. “5.5 Simple Harmonic Motion - Physics.” *OpenStax*, openstax.org/books/physics/pages/5-5-simple-harmonic-motion#:~:text=For%20small%20displacements%20of%20less,directly%20proportional%20to%20its%20displacement. Accessed 25 Aug. 2023.

approximation simplifies the mathematical analysis of the motion of the bifilar pendulum, granting the use of SHM equations.
Small angle approximation encompasses 3 fundamental relations:

$$\sin(\theta) \approx \theta$$

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2} \approx 1$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \approx \theta$$

For which θ is in radians (rads).

1.2.5 Derivation of the Factors Affecting the Period of Oscillation:

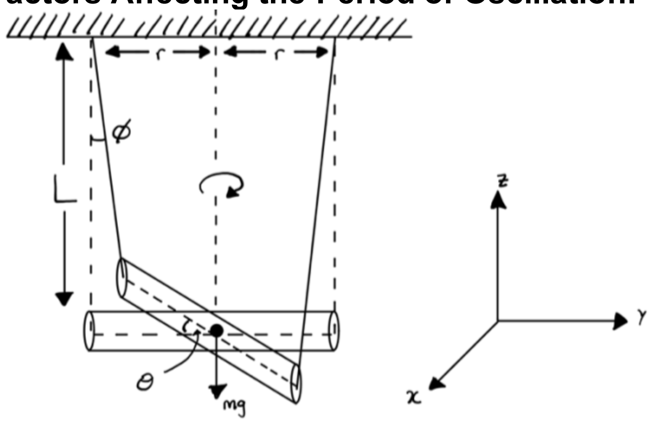


Figure 2: Bifilar Pendulum System Having Angular Displacement [Student's Own Work]

Figure 2 demonstrates the bifilar pendulum being displaced having some horizontal twist θ and vertical twist ϕ. There is no acceleration along the *z* axis. The vertical forces are therefore balanced.

The force acting downwards on the pendulum is therefore the force due to gravity, or its weight, denoted by mg. To counteract this force, with the assumption that the strings are identical, when the system is in its equilibrium position, the tension in each filar can be divided and denoted by $\left(\frac{mg}{2}\right)$. When twisted, this is altered to:

$$\left(\frac{mg}{2}\right)\cos(\phi) \approx \left(\frac{mg}{2}\right)$$

Figure 3 below presents another view of the twisted system considering just one filar and its effect on the cylindrical rod.

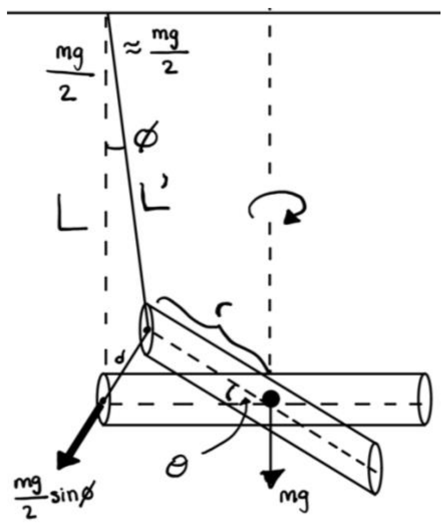


Figure 3: Filar Suspension Geometry [Student's Own Work]

L' is the twisted length of the filar relative to the front view of the system. In a small angle, L' and L are approximated to be the same value.
It can be seen that:

$$Lsin(\phi) = rsin(\theta) = d$$

d denotes the displacement in the horizontal plane.

Furthermore:

$$\tan(\theta) \approx \sin(\theta) \approx \theta \approx \frac{d}{r}$$
$$\therefore d \approx r\theta \qquad (1)$$

When the displaced pendulum is released, its motion is directed back towards its equilibrium position. This restoring force originates from the horizontal component of the tension force:

$$-\frac{mg}{2}\sin(\phi)$$

The direction of this force acting on the pendulum is shown by the thick arrow in Figure 3.

The negative sign exists because the force acting on the rod to bring it back to equilibrium position is in the opposite direction of the displacement of the rod.

The restoring force is caused by torque, affected by the magnitude of the force applied as well as the perpendicular distance between the point where the torque is applied and the point of application of the force (Guelph *Torque and Rotational Motion tutorial*).

$$\tau = F \cdot rsin(\theta)$$ [3]

Applying this to Figures 2 and 3, where the force acting on the object is the horizontal component of the tension force, along a perpendicular distance r, the torque force acting on each half of the suspended rod is modeled by:

[3] "Torque and Rotational Motion Tutorial." *Physics*, www.physics.uoguelph.ca/torque-and-rotational-motion-tutorial.

$$\tau_{half} = -\left(\frac{mg}{2}\right) rsin(\phi)$$

Because the torque acting on each side is towards the same direction, both halves can be added together to give a resulting torque force acting on the rod:

$$\tau = -2\left(\frac{mg}{2}\right) rsin(\phi) = -mgrsin(\phi) \qquad (2)$$

Newton's Second Law of Angular Motion States that "if more than one torque acts on a rigid body about a fixed axis, then the sum of the torques equals the moment of inertia times the angular acceleration" (10.7 Newton's Second Law for Rotation, OpenStax.org).

$$\tau_{net} = \sum_i \tau_i = I\alpha = I\left(\frac{d^2\theta}{dt^2}\right) \qquad (3)$$ [4]

I is the moment of inertia and α is the angular acceleration.

Both equations are represented by different angles; Equation 2 and 3 are in terms of ϕ and θ respectively. Thus, geometrical relationships observed must be utilized to identify a relationship between these angles to form an equation for the angular motion in terms of one angle.
From Figure 3:

$$d = Lsin(\phi) \qquad (4)$$

Combining equations 1 and 4 in terms of d,

$$r\theta \approx Lsin(\phi)$$
$$sin(\phi) \approx \left(\frac{r}{L}\right)\theta \qquad (5)$$

Substituting Equation 5 into Equation 2:

$$\tau = -mgrsin(\phi) = -mgr\left(\frac{r}{L}\right)\theta = -\frac{mgr^2}{L}\theta$$

Setting the above equation for the torque forces equal to Equation 3:

$$I\left(\frac{d^2\theta}{dt^2}\right) = -\frac{mgr^2}{L}\theta$$
$$\left(\frac{d^2\theta}{dt^2}\right) = -\frac{mgr^2}{IL}\theta$$
$$\frac{d^2\theta}{dt^2} + \left(\frac{mgr^2}{IL}\right)\theta = 0 \qquad (6)$$

The differential equation representing the motion of a simple pendulum undergoing SHM is:

[4] Moebs, William, et al. "10.7 Newton's Second Law for Rotation - University Physics Volume 1." *OpenStax*, openstax.org/books/university-physics-volume-1/pages/10-7-newtons-second-law-for-rotation. Accessed 01 Sept. 2023.

$$\frac{d^2\theta}{dt^2} + \left(\frac{g}{L}\right)\theta = 0$$ [5]

$$\omega = \sqrt{\frac{g}{L}} = \frac{2\pi}{T}$$

$$\therefore\ \omega^2 = \frac{g}{L} = \left(\frac{2\pi}{T}\right)^2$$

Thus,

$$\frac{d^2\theta}{dt^2} + (\omega^2)\theta = 0 \qquad (7)$$

Comparing Equations 6 and 7,

$$\omega^2 = \frac{mgr^2}{IL}$$

$$\left(\frac{2\pi}{T}\right)^2 = \frac{mgr^2}{IL}$$

$$\frac{4\pi^2}{T^2} = \frac{mgr^2}{IL}$$

$$\frac{T^2}{4\pi^2} = \frac{IL}{mgr^2}$$

$$T^2 = 4\pi^2\frac{IL}{mgr^2}$$

$$T = 2\pi\sqrt{\frac{IL}{mgr^2}}$$

$$\boldsymbol{T = \frac{2\pi}{r}\sqrt{\frac{IL}{mg}}} \qquad (8)$$

Whereby:
- T is the period of oscillation (s)
- r is the distance between the two strings (m)
- I is the moment of inertia (kgm^2)
- L is the length of the string suspending the rod (m)
- m is the mass of the rod (kg)
- g is the acceleration due to gravity (ms^{-2})

This relationship illustrates the effect that the distance between the strings, the mass of the rod, the moment of inertia, and the length of the string has on the period of the bifilar pendulum's oscillation.
- $T \propto \frac{1}{r}$
- $T \propto \sqrt{L}$

[5] More, Hemant. "Simple Harmonic Motion: Its Defining and Differential Equation." *The Fact Factor*, 18 Apr. 2020, thefactfactor.com/facts/pure_science/physics/simple-harmonic-motion/5540/. Accessed 01 Sept. 2023

- $T \propto \sqrt{I}$
- $T \propto \sqrt{\frac{1}{m}}$

1.2.6 Theoretical Considerations

For this investigation, several assumptions must be made to obtain a solid theoretical grasp of the occurring phenomenon:

- The rod is uniform throughout its length.
  - Ensures that the center of mass of the rod is in the center of the rod.
- Each string connected to the rod must be the same distance away from the center of mass as the other.
  - This ensures symmetry across the system, allowing for the rod to be lying flat.
- The pendulum system is simple (a mass is attached to a massless, inextensible string, and is suspended from a fixed support)
  - Nylon has negligible mass relative to the rod and exhibits minimal elastic stretch under the load applied.
  - This eliminates the need to factor in the mass of the string.
- The initial angular displacement of the rod is kept small
  - Small angle approximation simplifies the theory behind the system's motion.
- The pivots (or base) from which the pendulums oscillate are frictionless
  - The string is tied directly to a fixed overhead support with no bearing or axle, eliminating mechanical friction at the pivot.
  - Ensures that the motion of the system is dictated solely by the torque force.
- The pendulum has a proper back-and-forth path of oscillation, not an elliptical path
  - This pendulum works under the principles of SHM.
- The system assumes there is no air resistance or external forms of damping.
  - At angular velocities below $0.5\ \frac{rad}{s}$, aerodynamic drag is several orders of magnitude smaller than the restoring torque.
  - This ensures that the system conserves its mechanical energy during the entirety of its motion.

1.3 Objective

The primary objective of this investigation is to explore the relationship of changes to the distance between the strings suspending a bifilar pendulum and the resulting effect in the period of oscillation. This relationship involves identifying whether the relationship is linear, exponential, or a different pattern altogether. Using this acquired relationship, a value can be derived for the moment of inertia of the suspended rod about its central axis.

1.4 Hypothesis

Based on the principles of mechanics and the behavior of oscillatory systems, it is hypothesized that there exists an inverse relationship connecting the distance (r) between the strings suspending the bifilar pendulum and its period of oscillation (T).

This hypothesis is due to the principles of torque force, whereby an increase in the distance between the strings increases the torque acting on the rod, which in turn results in a faster motion and lower period of oscillation. Through the upcoming investigation, this hypothesis can be accordingly accepted or rejected.

2.1 Methodology

2.1.1 Variables

Independent Variable

- Distance between the two strings suspending the rod (Each string must be an equal distance away from the center of mass)

Dependent Variable

- Time period of oscillation of the bifilar pendulum measured through stopwatch

Controlled Variables

- Mass, Length, Material of the rod, and Length of String suspending the rod
    - Affects the period of oscillation as well as the moment of inertia of the rod
    - Length was measured before every trial. Steel rod was used
- Support structure
    - Affects the twisting of the rod
    - Structure used as in Figure 1 below
- Starting displacement
    - 10° initial displacement; displacement affects the twisting angle of the bifilar pendulum, which would affect the time of oscillation.
    - Released from the same marked point on a wall
- String material
    - Same string throughout the experiment was used
    - Changing the material of the string would affect the Young's Modulus of Elasticity

2.1.2 Apparatus

- Cylindrical (hollow) rod; Quantity: 1
- Solid base for hanging; Quantity: 1
- Nylon string; Length: 2 meters
- Digital stopwatch; Quantity: 1; Uncertainty: ± 0.01 seconds
- Measuring tape; Quantity: 1; Uncertainty: ± 0.0005 meters
- Marker; Quantity: 1
- Scissors; Quantity: 1

2.1.3 Procedure

1. Measure the length of the rod from one end to the other and mark the midpoint of the rod
2. Mark different points on both sides of the center. This indicates where the strings will be placed in different trials; ensure that the distance from the left string to the center is the same as the distance from the right string to the center
3. Find a suitable place to act as a pivot to hang the strings and rod (ie. retort stand, curtain rod)

4. Tightly tie two strings of equal length to the pivot from Step 3
5. From the point on Step 4, cut out a piece of nylon string that goes beyond the intended value for the length of the string (ie. if the desired length of the string is 70 cm, cut the string to be 85-90 cm)
6. Mark the point on the string which is the desired length (ie. 70 cm) from the top pivot
7. Tie the bottom end of both the strings to the rod, ensuring that the marked position from Step 6 is located just on top or along the rod.
8. Ensure that both the strings are an equal distance apart from the center of mass as well as from the top pivot by using a measuring tape
9. Mark a constant point from which the rod will be displaced for each trial
10. Hold the rod at the position of displacement from Step 9 and calibrate a stopwatch
11. Start the stopwatch simultaneously as the rod is released
12. Measure the time taken for 10 total oscillations
13. Record the data for the time and bring the rod to rest.
14. Repeat Steps 10-13 four more times to get 5 trials.
15. Repeat Steps 10-14 seven more times for a total of 8 different independent variables.

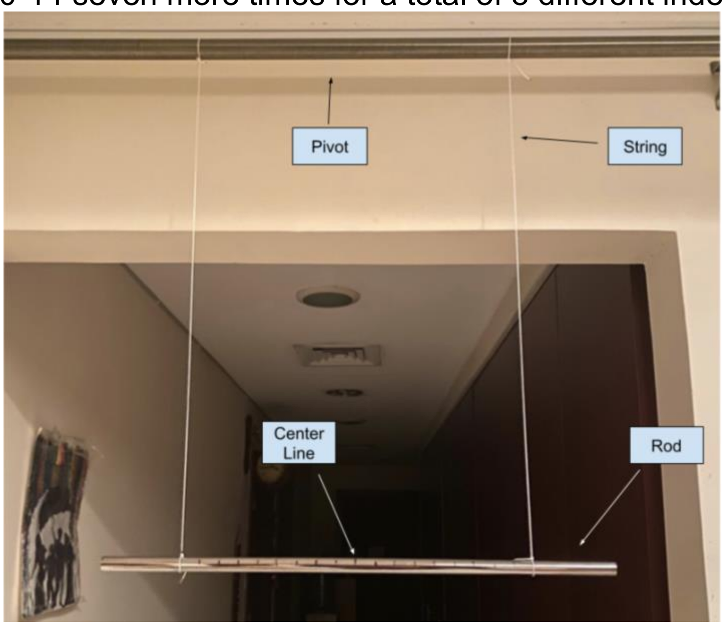


Figure 1: Bifilar Pendulum Setup

This investigation does not have any key safety, ethical, or environmental considerations.

Rod Mass	Rod Length	String Length
$0.14603 \pm 0.00001\ kg$	$0.655 \pm 0.0005\ m$	$0.700 \pm 0.0005\ m$

3.1 Raw Data

Distance Between Two Strings (m) / ± 0.0005 m	Time for 10 Oscillations (s)				
	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5
0.1000	61.90	62.27	62.43	62.02	62.18
0.1500	41.49	41.42	41.56	41.44	41.58
0.2000	31.44	31.16	31.37	31.25	31.09
0.2500	24.81	25.07	24.87	25.05	25.00
0.3000	20.82	20.86	20.80	20.80	20.84
0.3500	17.91	17.78	17.76	17.80	17.67
0.4000	15.76	15.77	15.70	15.81	15.70
0.4500	13.74	13.90	13.84	13.83	13.77

Table 1: Raw Data Table

3.2 Processed Data

Distance Between Two Strings (m) / ± 0.0005 m	Avg Time for 10 Oscillations (s)	Time Period (s)	Uncertainty in Average Time for 10 Oscillations (s)	Uncertainty in Average Time Period (s)
0.1000	62.16	6.22	± 0.27	± 0.03
0.1500	41.50	4.15	± 0.08	± 0.01
0.2000	31.26	3.13	± 0.18	± 0.02
0.2500	24.96	2.50	± 0.13	± 0.01
0.3000	20.82	2.08	± 0.03	± 0.00
0.3500	17.78	1.78	± 0.12	± 0.01
0.4000	15.75	1.57	± 0.06	± 0.01
0.4500	13.82	1.38	± 0.08	± 0.01

Table 2: Processed Data Table

Highlighted cells are present in sample calculations.

3.2.1 Sample Calculations for Data and Uncertainties

a) Average time for 10 Oscillations of Bifilar Pendulum

$$= \frac{Trial\ 1 + Trial\ 2 + Trial\ 3 + Trial\ 4 + Trial\ 5}{5}$$

Calculations for 0.1000 m:

$$= \frac{61.90 + 62.27 + 62.43 + 62.02 + 62.18}{5}$$
$$= 62.16\ s$$

b) Average Time Period of Bifilar Pendulum

Calculations for 0.1000 m:

$$= \frac{62.16}{10}$$
$$= 6.216 \approx 6.22\ s$$

c) Uncertainty in Average Time for 10 Oscillations

$$= \pm\left(\frac{Maximum\ Time\ for\ 10\ Oscillations - Minimum\ Time\ for\ 10\ Oscillations}{2}\right)$$ [6]

Calculations for 0.1000 m:

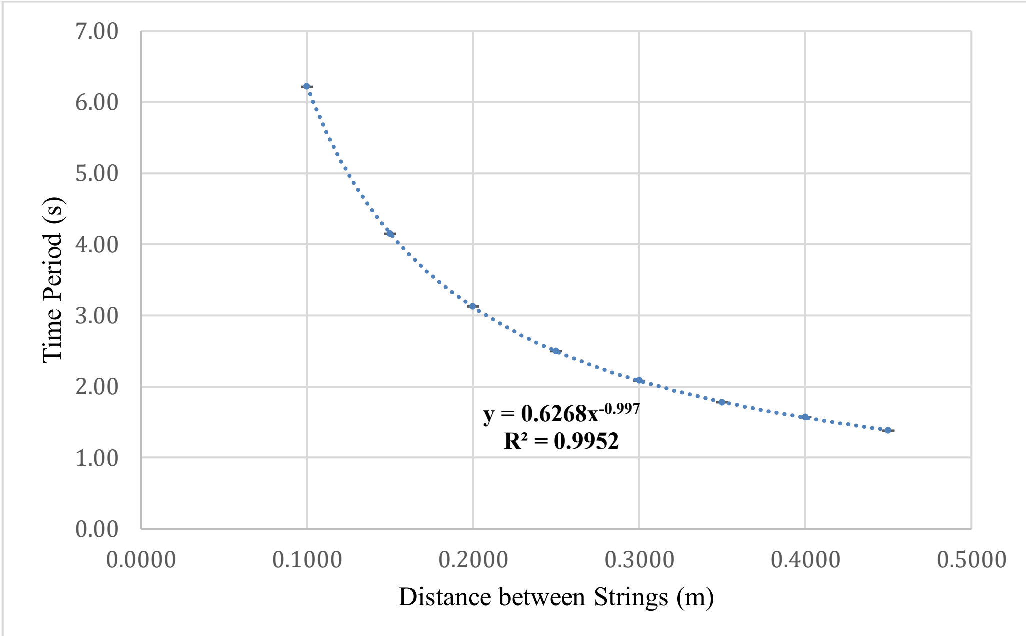
$$= \pm\left(\frac{62.43 - 61.90}{2}\right)$$
$$= \pm 0.265 \approx \pm 0.27\ s$$

d) Uncertainty in Time Period of Oscillation of Bifilar Pendulum

Calculations for 0.1000 m:

$$= \pm\left(\frac{0.265}{10}\right)$$
$$= \pm 0.0265 \approx \pm 0.03\ s$$

[6] Irfansyahril. “PHY C2: Errors & Uncertainties.” *ProDuckThieves*, 6 July 2019, produckthieves.home.blog/2019/01/27/phy-c2-errors-uncertainties/. Accessed 2 Sep. 2023.



Graph 1: Time Period of Oscillation for Bifilar Pendulum vs Distance Between Strings

3.2.2 Trends

Graph 1 shows a negative association between the period of oscillation of the bifilar pendulum and the distance between the two strings suspending the rod from the pivot. For an increase in the distance between the two strings, the period decreases at an exponential rate. The computer, using Microsoft Excel, returns a modeled function, given by:

$$T(r) = 0.6268r^{-0.997}$$

Where:

- $T(r)$ is the period of the bifilar pendulum (seconds)
- r is the distance between the two strings suspending the rod (meters)
    - Here, r is not from the string to the center.

The relationship can be interpreted by exploring the torque acting on the bifilar pendulum system.

$$\tau = Frsin(\theta)$$

The two equations previously derived for torque can be utilized:

$$\tau = -mgrsin(\phi)$$

$$\tau_{half} = -\left(\frac{mg}{2}\right)rsin(\phi)$$

The forces acting on the system are the force due to gravity, the tension due to the string, and torque force, of which the force due to gravity and tension are kept being constant. The torque acting on the rod causing it to rotate about its rotational axis is proportional to the distance between the two strings. As the distance between the strings increase, there is an increase in the perpendicular distance on which the force acts, and therefore by the above relationship, the torque force acting on the rod increases. This increase in torque implies that the overall motion is faster, indicating that oscillations take less time, signifying a decreased period for the bifilar pendulum system. Contrarily, a decrease in the distance between the strings would indicate a lower torque force, thus decreasing the overall velocity of the rod during each cycle, resulting in oscillations taking more time and a thus higher period.

3.3 Linearization

The formula derived for the relationship between the period of the pendulum and the distance between the two strings, once linearized, can be used to find the moment of inertia of the pendulum.

$$T = \frac{2\pi}{r}\sqrt{\frac{IL}{mg}}$$

$$T = \left(2\pi\sqrt{\frac{IL}{mg}}\right) * \frac{1}{r}$$

This equation is in the form of a linear equation: $y = kx$

Graphing a $T \; vs \frac{1}{r}$ plot (where $y = T$ and $x = \frac{1}{r}$) produces a best fit line that passes through the origin, having a gradient:

$$k = 2\pi\sqrt{\frac{IL}{mg}} \qquad (9)$$

PROPOGATION OF UNCERTAINTIES OF LINEARIZED GRAPH			
Time Period (s)	**1/Distance Between the Strings (m^{-1})**	**Fractional Uncertainty in Distance Between the Strings**	**Absolute Uncertainty in Reciprocal of Distance Between the Strings (m^{-1})**
6.22	**10.0000**	**0.0050**	**± 0.0500** $\approx 5 \cdot 10^{-2}$
4.15	**6.6667**	**0.0033**	**± 0.0222** $\approx 2 \cdot 10^{-2}$
3.13	**5.0000**	**0.0025**	**± 0.0125** $\approx 1 \cdot 10^{-2}$
2.50	**4.0000**	**0.0020**	**± 0.0080** $\approx 8 \cdot 10^{-3}$
2.08	**3.3333**	**0.0017**	**± 0.0056** $\approx 6 \cdot 10^{-3}$
1.78	**2.8571**	**0.0014**	**± 0.0041** $\approx 4 \cdot 10^{-3}$
1.57	**2.5000**	**0.0013**	**± 0.0031** $\approx 3 \cdot 10^{-3}$
1.38	**2.2222**	**0.0011**	**± 0.0025** $\approx 2 \cdot 10^{-3}$

Table 3: Processed Data Table for Linearized Data

Highlighted cells are present in sample calculations.

3.3.1 Sample Calculations for Data and Uncertainties

a) Reciprocal of distance between strings $\left(\frac{1}{r}\right)$

Calculations for 0.1000 m:

$$= \frac{\mathbf{1}}{\mathbf{0.1000}} = \mathbf{10.0000}\ (\boldsymbol{m^{-1}})$$

b) Fractional uncertainty in distance between strings

$$= \frac{\textbf{\textit{Absolute Uncertainty in Distance Between Strings}}}{\textbf{\textit{Measured Value of Distance Between Strings}}}$$

Calculations for 0.1000 m:

$$= \frac{\mathbf{0.0005}}{\mathbf{0.1000}} = \mathbf{0.0050}$$

c) Absolute Uncertainty in Reciprocal of Distance Between the Strings (m^{-1})

$$= (\textbf{\textit{Fractional Uncertainty in Distance Between Strings}}) \cdot (\textbf{\textit{Measured Value of Reciprocal of Distance Between the String}})$$

Calculations for 0.1000 m:

$$= \pm\mathbf{0.0050} \cdot \mathbf{10} = \pm\mathbf{0.0500}\ \boldsymbol{m^{-1}}$$

3.3.2 Calculations for Maximum and Minimum Gradients

a) For Maximum Gradient

$$x_1 = \mathbf{2.2222 + 0.0025 = 2.2247}$$
$$y_1 = \mathbf{1.38 - 0.01 = 1.37}$$

$$x_2 = \mathbf{10.0000 - 0.0500 = 9.9500}$$
$$y_2 = \mathbf{6.22 + 0.03 = 6.25}$$

$$Maximum\ Gradient = \frac{y_2 - y_1}{x_2 - x_1}$$

$$Maximum\ Gradient = \frac{6.25 - 1.37}{9.9500 - 2.2247} = 0.632\ metre\ seconds$$

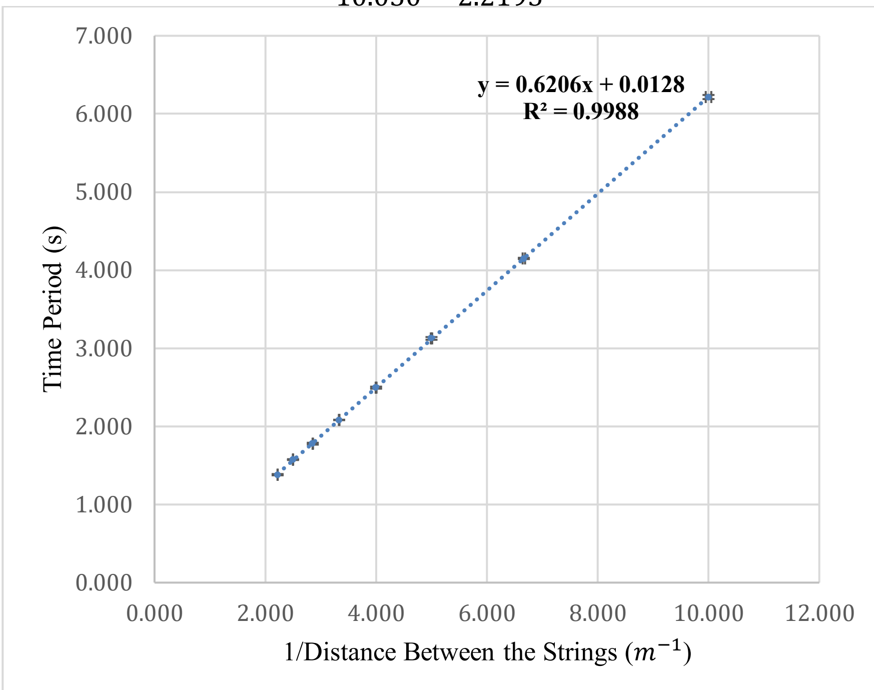
b) For Minimum Gradient

$$x_1 = \mathbf{2.2222 - 0.0025 = 2.2195}$$
$$y_1 = \mathbf{1.38 + 0.01 = 1.39}$$

$$x_2 = \mathbf{10.0000 + 0.0500 = 10.050}$$
$$y_2 = \mathbf{6.22 - 0.03 = 6.19}$$

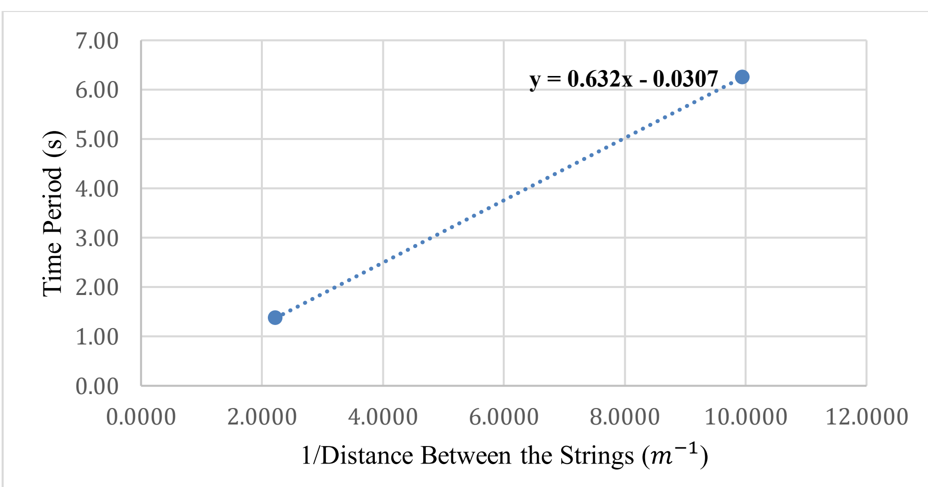
$$Minimum\ Gradient = \frac{y_2 - y_1}{x_2 - x_1}$$

$$Minimum\ Gradient = \frac{6.19 - 1.39}{10.050 - 2.2195} = 0.613\ metre\ seconds$$

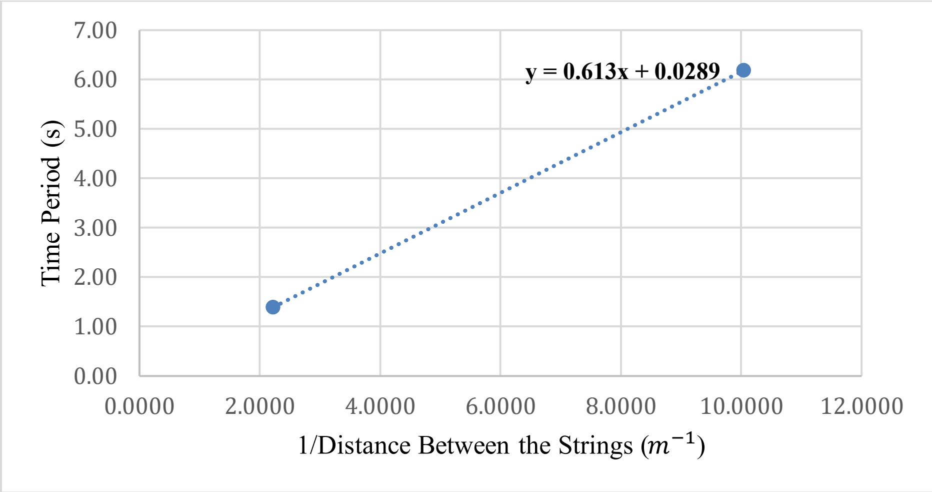


Graph 2: Time Period of Oscillation of Bifilar Pendulum v/s 1/Distance Between the Strings

The error bars in Graph 1 and Graph 2 are not visible as they are very small because of the low random uncertainties in the measuring instruments.The best-fit line yields a small positive y-intercept, compared to the theoretical prediction of zero. This minor offset likely reflects residual systematic error from manual timing or slight non-linearity at large string separations where the small-angle approximation begins to break down. At less than 1% of the smallest period measured, the intercept does not materially affect the gradient-derived MoI.



Graph 3: Maximum Gradient for Linearized Graph



Graph 4: Minimum Gradient for Linearized Graph

Minimal discrepancies to the maximum and minimum gradients are indicative of slight imprecisions to the data attained.

3.4 Deriving Moment of Inertia for Hollow, Cylindrical Rod

From the linearized data, with $T = \left(2\pi\sqrt{\frac{IL}{mg}}\right) * \frac{1}{r}$, the mean gradient of the linearized graph is:

$$\frac{Gradient_{max} + Gradient_{min}}{2}$$

$$= \frac{0.632 + 0.613}{2}$$

$$\boldsymbol{Mean\ Gradient} = 0.6225\ metre\ seconds$$

From Equation 9,

$$2\pi\sqrt{\frac{IL}{mg}} = Mean\ Gradient = \ 0.6225\ metre\ seconds$$

With $L = 0.700m$ and $m = 0.14603kg$,

$$2\pi\sqrt{\frac{I(0.700)}{(0.14603)(9.81)}} = 0.6225$$

Rearranging for I,

$$I = \left(\frac{0.6225}{2\pi}\right)^2 \cdot \frac{(0.14603)(9.81)}{(0.700)}$$

$$I = 0.02008778 \approx 2.01 \cdot 10^{-2}\ kgm^2$$

Absolute Uncertainty for Moment of Inertia (MoI)

Absolute Uncertainty for MoI

$$= \pm\,(Fractional\ Uncertainty\ for\ MoI\,)\cdot(Measured\ Value\ for\ MoI) \qquad (10)$$

Fractional Uncertainty for MoI

$$= (2\,\cdot Fractional\ Uncertainty\ in\ Gradient) + (Fractional\ Uncertainty\ in\ Mass) + (Fractional\ Uncertainty\ in\ Length) + (Fractional\ Uncertainty\ in\ Acceleration\ due\ to\ Gravity)$$

Fractional uncertainty in acceleration due to gravity is neglected because $g = 9.81\ ms^{-2}$ is directly taken from the Physics Data booklet.

$$\frac{\Delta MoI}{MoI} = 2\frac{\Delta k}{k} + \frac{\Delta m}{m} + \frac{\Delta L}{L} \qquad (11)$$

$$Uncertainty\ in\ Gradient\ (\Delta k) = \frac{Gradient_{max} - Gradient_{min}}{2}$$

$$= \frac{0.632 - 0.613}{2}$$

$$\boldsymbol{Absolute\ Uncertainty\ in\ Gradient} = \pm\,0.0095\ metre\ seconds$$

From Equation 11,

$$\frac{\Delta MoI}{MoI} = 2\frac{0.0095}{0.6225} + \frac{0.00001}{0.14603} + \frac{0.0005}{0.700} = 0.03130485$$

From Equation 10,

$$Absolute\ uncertainty\ for\ MoI = \pm\,(0.03130485)\cdot(0.02008778)$$

$$Absolute\ uncertainty\ for\ MoI = \pm\,0.00062884\ \approx 6.29\cdot 10^{-4} kgm^2$$

The final measured value for the Moment of Inertia for the hollow, cylindrical rod is:

$$\mathbf{2.01\cdot 10^{-2} \pm\ 6.29\cdot 10^{-4}}\boldsymbol{kgm^2}$$

The percentage error for the MoI is:

$$= \frac{6.29\cdot 10^{-4}}{2.01\cdot 10^{-2}}\cdot 100$$

$$= 3.12935323 \approx \mathbf{3.13}\%$$

3.5 Comparison with Theoretical Value

For the hollow cylindrical rod used in this investigation, the theoretical moment of inertia about the central perpendicular axis is:

$$I = m\left[\frac{1}{4}(r_1^2 + r_2^2) + \frac{1}{12}l^2\right]$$

The radii r_outer and r_inner were not recorded during the investigation. However, for any physically plausible dimensions of this rod, the radial term $\frac{1}{4}(r_1^2 + r_2^2)$ contributes less than 2% of the theoretical MoI. This is because the length term dominates entirely: with $l = 0.655\ m$, the l^2 contribution is approximately $0.429\ m^2$, while even a generous outer radius of $0.040\ m$ gives $r^2 = 0.0016\ m^2$, a factor roughly 270 times smaller. The theoretical value reduces to:

$$I_{theory} \approx \frac{1}{12}ml^2 = \frac{1}{12}(0.14603)(0.655)^2 = 5.22 \times 10^{-3}\ kg \cdot m^2.$$

This is approximately four times smaller than the experimentally derived value of $2.01 \times 10^{-2}\ kgm^2$. This discrepancy warrants consideration. The bifilar method is sensitive to any motion that is not purely torsional: elliptical paths of oscillation, swinging rather than twisting, and unequal string lengths all inflate the effective period and therefore inflate the derived MoI. Given that elliptical motion was identified as a limitation in Section 4.1.2, this is the most plausible source of the systematic overestimation. A more controlled setup (ie. ensuring purely torsional oscillation through a rigid guide or video tracking) would be expected to bring the experimental value closer to the theoretical prediction.

4.1 Evaluation

4.1.1 Strengths of Investigation

The modelled function, $T(r)$, has an R^2 value of 0.9952, while the linearized T vs $\frac{1}{r}$ graph achieves an R^2 of 0.9988, indicating that the regression prediction almost perfectly fits the data set. This precision was due to several precautionary measures.[7]

Using a nylon string was significant to this investigation due to its low friction (minimizing external energy loss of the pendulum), low flexibility (allowing reductions in extra vibrations of the pendulum), and high durability (allowing multiple trials without degradations). The thin threads used ensured for greater accuracy as it was easier to navigate the center of the thread for calculation and processing purposes. Systematic errors in the experiment were minimized.

The measuring tape used has an uncertainty of ± 0.0005 meters; this small uncertainty indicates higher levels of precision and accuracy in the measurements of the distance between the strings.

Ensuring a constant small-angle displacement of $10°$ throughout the experiment reduced nonlinearities, impacts of air resistance, and harmonic contributions. Generally, many

[7] "R-Squared." *Financial Modelling Terms Explained*, www.causal.app/define/r-squared#:~:text=R%2Dsquared%20is%20a%20measure,1%20indicates%20a%20perfect%20fit.

mathematical relations derived by other researchers to investigate systems that exhibit oscillations and harmonic motion tend to factor in small angle as a means of establishing a linear and more precise relationship between oscillation and the changed factor.

Using a rigid support structure to carry the bifilar pendulum offered benefits to the precision of the data providing stable attachment points and a consistent positioning of the strings relative to its point of release.

4.1.2 Limitations and Sources of Error

A manual stopwatch was used rather than a photogate or video analysis system. Human reaction time introduces a timing error at the start and stop of each measurement. Assuming a reaction time of $0.20\ s$ per trigger event, the total timing error per trial is $2 \times 0.20 = 0.40s$. As a percentage of the measured duration this ranges from $\frac{0.40}{62.16} \times 100 = 0.64\%$ for the longest run ($r = 0.10\ m$) to $\frac{0.40}{13.82} \times 100 = 2.89\%$ for the shortest run ($r = 0.45\ m$). The worst-case contribution of 2.89% is within the dominant gradient uncertainty of 3.13%, confirming that reaction time is a secondary error source. Nevertheless, video analysis at 60 fps would eliminate this error entirely and is recommended for future work.

For the initial angular displacement of the bifilar pendulum, a point on a wall was marked from which the pendulum was released. There may have been scope for a parallax error, whereby the desired angular displacement may not have been attained as anticipated. The imprecise angular release could introduce systematic errors that affect the data. Utilizing a digital protractor could mitigate the parallax errors of this investigation.

During the motion of the bifilar pendulum system, the rod also had an elliptical path of motion at times. This caused systematic errors since there was a non-uniform motion of the rod during the oscillations. Stabilizing the setup (ensuring it comes to a complete halt prior to a new trial) and ensuring a controlled trial each time would help mitigate any fluctuations caused by the shaking effect.

The solid rod to which the string is attached may be susceptible to added frictional forces acting on it, affecting its period by contributing to additional energy loss. Additionally, the friction between the string and the rod could have affected the data. The use of lubrication or low-friction materials such as Teflon could help lessen the systematic errors caused by friction.

At times, during changes to the distance between the strings, there may have been some inaccurate period measurements due to the imbalance in suspension and mass distribution of the system. The stability of the system can be enhanced by utilizing a spirit level to ensure that the cylindrical rod is perfectly horizontal and parallel to the ground. By ascertaining the parallel structure of the rod, the measurements for the lengths calculated by the rulers are more accurate, thus helping reduce random errors in the apparatus.

4.2 Conclusion

This investigation explores how the period of oscillation can be used to determine the moment of inertia of a hollow, cylindrical rod. The hypothesis formulated can be accepted, whereby an increase in the distance between the strings decreases the period of oscillation. The period of oscillation of the bifilar pendulum with respect to changes in the distance between the strings can be modelled by the function:

$$T = \left(2\pi \sqrt{\frac{IL}{mg}}\right) * \frac{1}{r}$$

Using a cylindrical rod of length 0.650 m, having a mass of 0.14603 kg, and being suspended by strings a length of 0.700 m, and furthering it by utilizing the mean gradient of 0.6225, the moment of inertia was concluded to be $2.01 \cdot 10^{-2} \pm 6.29 \cdot 10^{-4}\ kgm^2$.

This is the moment of inertia of the hollow, cylindrical rod about its center. The internal consistency of the experimental method is evidenced by the low percentage error of 3.13% in the gradient-derived MoI. The experimental value of $I = 2.01 \cdot 10^{-2} \pm 6.29 \cdot 10^{-4}\ kgm^2$ is approximately four times larger than the theoretical thin-rod approximation of $5.22 \cdot 10^{-3}\ kgm^2$, a discrepancy attributed primarily to non-purely torsional oscillation as discussed in Section 3.5. Despite this, the internal consistency of the bifilar method is demonstrated by the $R^2 = 0.9988$ linearised fit and 3.13% gradient uncertainty.

The principal strength of the technique is the determination of MoI without prior knowledge of the object's internal geometry, making it directly applicable to objects of complex or unknown shape such as aircraft components. Future work should focus on eliminating non-torsional motion through mechanical constraints and replacing manual timing with automated video analysis.

4.3 Future Investigations

This investigation can be extended primarily by altering any of the other variable parameters of the system (i.e. mass of rod, length of string).

Utilizing an elastic material having a higher Young's Modulus (E) value for the object suspending the rod could also be investigated, since the rotation for the bifilar pendulum system would result in the rotation of the base as well, reducing the torque force and thus altering the overall dynamics of the system. This change reduces the overall shaking of the system, resulting in a more accurate depiction of its actual. Such a system is widely seen with suspension bridges and is a fundamental construct of modern-day architecture and infrastructure.

Affecting the shape of the bifilar pendulum being suspended (i.e. using a rectangular or triangular shaped-rod) could also be investigated. The period of oscillation would be influenced by changes to the shape since different shapes have varying moments of inertia. Investigating this could set an intriguing investigation that has profound relevance to real life scenarios should one choose to delve deep into non-uniform shapes.

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