



The Role of Eigenvalues and Eigenvectors in the Qualitative Behavior of Linear Systems

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Abstract

This paper studies the system $x'(t) = Ax(t)$ and explains how the eigenvalues and eigenvectors of A determine the qualitative shape of its solutions. The aim is to provide a unified algebraic and geometric account of how changes in a matrix alter stability and phase-portrait type. The analysis combines the matrix exponential, diagonalization, eigenvector directions, the trace-determinant plane, and controlled parameter comparisons.

The main focus is on real two-dimensional linear systems. Real eigenvalues produce growth or decay along eigendirections; complex eigenvalues describe oscillation with an exponential envelope determined by their real part. A solved Jordan-block example shows how equal eigenvalues can produce different phase-portrait geometry. In the complex region, varying trace at fixed determinant changes both the common real part and the oscillation frequency. At fixed trace, varying determinant can change stability across a zero-eigenvalue boundary and geometry across a repeated-eigenvalue boundary. Trace also determines the exponential rate of area change, which must be distinguished from convergence of individual trajectories. The comparisons illustrate classical theory: eigenvalues determine exponential rates, while eigenspaces and Jordan structure complete the description.



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1. Introduction

A linear system of differential equations describes how several quantities change at the same time. Such models appear in mechanics, electrical engineering, economics, biology, and many other fields. A common form of a linear dynamical system is

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t),$$

where $\mathbf{A}$ is a constant matrix and $\mathbf{x}(t)$ represents the state of the system at time t .

For this paper, the main question is what happens to $\mathbf{x}(t)$ as t becomes large. Depending on the structure of the matrix $\mathbf{A}$, solutions may approach an equilibrium, move away from it, or rotate around it. The solution of the system can be written using the matrix exponential:

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0).$$

However, this formula alone does not explain why different matrices produce different types of behavior. To answer this question, it is necessary to examine the eigenvalues and eigenvectors of the matrix.

This paper addresses the following research question:

How do the eigenvalues and eigenvectors of a matrix influence the qualitative behavior of solutions to the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$?

To answer this question, the paper first introduces the main mathematical tools: the matrix exponential, eigenvalues, eigenvectors, and diagonalization. It then classifies the behavior of two-dimensional linear systems using real and complex eigenvalues. Finally, it examines how changes in matrix parameters, especially the trace and determinant, affect the phase portrait and the long-term behavior of solutions.

1.1 Related Literature and Scope

Classical treatments of linear differential systems develop the matrix exponential, spectral decomposition, and phase-plane classification from complementary viewpoints. Strang (2010, 2016) emphasizes diagonalization and $\exp(\mathbf{A}t)$; Boyce et al. (2017), Hirsch et al. (2012), Strogatz (2015), Perko (2001), and Teschl (2012) connect eigenvalue configurations to stability and planar phase portraits. Higham (2008) treats the matrix exponential as a matrix function and discusses computational issues that arise beyond small hand calculations.

The trace–determinant diagram used here is a standard coordinate-free summary for real 2×2 systems (MIT OpenCourseWare, 2011a, 2011b; Strogatz, 2015). However, trace and determinant determine only the characteristic polynomial. Repeated eigenvalues require eigenspace or Jordan information, and nonnormal matrices can exhibit substantial finite-time amplification even when all eigenvalues have negative real part (Trefethen & Embree, 2005). Recent work extends this perspective to large networked systems, showing how network topology shapes transient responses near stable equilibria (Tarnowski et al., 2020).

This paper therefore presents a focused synthesis and controlled theoretical-computational comparison rather than a new theorem. It compares matrices with the same eigenvalues but different eigenvectors, analyzes a repeated-eigenvalue Jordan example, and studies two parameter families that separate the effects of trace and determinant. The generated phase portraits connect the algebraic conditions to observable trajectory geometry.

Computational details. The phase portraits in Figures 1, 3, 5a–5c, and 6 were generated with Python, NumPy, SciPy, and Matplotlib. Trajectories were evaluated as $x(t) = \exp(At)x(0)$, using `scipy.linalg.expm` at 601 equally spaced times from 0 to 6. Unless stated otherwise, eight initial points lie on the circle of radius 2.4 at angles $\pi/8 + k\pi/4$, $k = 0, \dots, 7$. The center panel instead uses initial points (0.8, 0), (1.5, 0), and (2.2, 0). Each panel uses equal axis scales on $[-3, 3]^2$; portions outside the plotting window are clipped. The vector field is sampled on a 19×19 grid. Its arrows are normalized to show direction, not speed, and trajectory arrows indicate increasing time. Dashed lines mark the invariant directions discussed in the text. Figures 2 and 4 are analytical diagrams of $D = T^2/4$ and of the parameter path, respectively.

2. Mathematical Background

This section discusses the matrix exponential, eigenvalues, eigenvectors, and diagonalization.

2.1 Matrix Exponential

Consider $x'(t) = Ax(t)$, where A is a constant real $n \times n$ matrix and $x(0) = x_0$. The scalar equation $x' = ax$ has solution $\exp(at)x_0$. Replacing the scalar a by A leads to the matrix exponential, defined by a power series:

$$\begin{aligned} e^{At} &= \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}, & e^{A0} &= I, \\ \frac{d}{dt} e^{At} &= \sum_{k=1}^{\infty} \frac{A^k t^{k-1}}{(k-1)!} = A e^{At}, \\ x(t) &= e^{At} x_0, & x'(t) &= A e^{At} x_0 = A x(t). \end{aligned}$$

Here I is the identity matrix and $A^0 = I$. The matrix power series and its termwise derivative converge uniformly on every bounded interval of t ; for example, their norms are bounded by convergent scalar exponential series. Term-by-term differentiation is therefore valid. The displayed calculation verifies the differential equation, and $\exp(A \cdot 0) = I$ verifies the initial condition.

By uniqueness for a linear initial-value problem with continuous coefficients, this is the solution for every initial vector x_0 (Boyce, DiPrima, & Meade, 2017). The formula gives the state at any time; the next section explains how eigenvalues and eigenvectors make its qualitative behavior transparent.

2.2 Eigenvalues, Eigenvectors, and Diagonalization

Eigenvalues are useful because they appear in the exponential solution. An eigenvector u is a nonzero vector such that Au is a scalar multiple of u . The line spanned by a real eigenvector is invariant; a negative eigenvalue reverses the vector's orientation under multiplication by A . Algebraically,

$$Au = \lambda u,$$

where u is the eigenvector and λ is the corresponding eigenvalue.

Rearranging this equation gives

$$Au - \lambda u = 0.$$

Since $u = lu$, this becomes

$$Au - \lambda lu = 0.$$

Therefore,

$$(A - \lambda I)u = 0.$$

For a nonzero solution, u , to exist, the matrix $A - \lambda I$ must be singular. Therefore,

$$\det(A - \lambda I) = 0.$$

This equation is called the characteristic equation of the matrix. The solutions of the characteristic equation are the eigenvalues of A .

Eigenvalues are important because they determine the exponential factors that appear in the solution of a linear system. If an eigenvalue has a negative real part, the corresponding part of the solution decays over time. If an eigenvalue has a positive real part, the corresponding part of the solution grows. If the eigenvalue has an imaginary part, rotation appears in the phase portrait.

A matrix A is diagonalizable if there exist an invertible matrix P and a diagonal matrix Λ such that

$$A = P\Lambda P^{-1}.$$

The columns of P are linearly independent eigenvectors of A , and the diagonal entries of Λ are the corresponding eigenvalues. This representation makes the exponential especially transparent:

$$e^{At} = P e^{\Lambda t} P^{-1}.$$

If A is diagonalizable, the change of coordinates $y = P^{-1}x$ transforms

$$x'(t) = Ax(t)$$

into

$$y'(t) = \Lambda y(t).$$

Each component then evolves independently as $y_i(t) = c_i e^{\lambda_i t}$. Consequently, the real parts of the eigenvalues determine growth or decay, while the eigenvectors determine the invariant directions in the original coordinate system (Strang, 2010; Strang, 2016).

A real 2×2 matrix with distinct real eigenvalues is diagonalizable over the real numbers. A real 2×2 matrix with a nonreal conjugate pair is diagonalizable over the complex numbers, and its real dynamics can be represented by a rotation-dilation block. With a repeated eigenvalue, the dimension of the eigenspace must be checked. Section 4.1 solves both the two-eigenvector and one-eigenvector cases.

3. Classification of Dynamic Behavior Through Eigenvalues

The examples in this section use simple matrices to isolate spectral and geometric effects. Invertible linear changes of coordinates preserve stability, although they may change angles, lengths, and eigendirections.

The origin is stable in the sense of Lyapunov if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that every solution starting within distance δ of the origin remains within distance ε for all $t \geq 0$. It is asymptotically stable if it is stable and all sufficiently nearby solutions also converge to the origin as t tends to infinity. It is unstable if the stability condition fails. A center is stable but not asymptotically stable.

This section classifies the behavior of two-dimensional linear systems of the form

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

using the eigenvalues of the matrix A . In each case, the signs and real parts of the eigenvalues determine whether solutions approach the origin, leave it, or rotate around it (Strogatz, 2015; Hirsch, Smale, & Devaney, 2012; MIT OpenCourseWare, 2011a).

For two-dimensional systems, it is useful to separate the analysis into two major cases: real eigenvalues and complex eigenvalues. In the real case, the signs of the eigenvalues determine the main behavior. In the complex case, the real and imaginary parts determine whether trajectories spiral inward, spiral outward, or rotate without growth or decay.

Eigenvectors also have a geometric role. For real eigenvalues, eigenvectors determine the directions along which solutions grow or decay. In a diagonal system, these directions are the coordinate axes. In a diagonalizable non-diagonal system, the eigendirections may be tilted, but they still describe the directions associated with the eigenvalues. Thus, eigenvalues determine the rate and type of behavior, while eigenvectors determine the directions in which this behavior appears.

3.1 Systems with Real Eigenvalues

First, consider systems whose eigenvalues are real numbers. In this case, the behavior can be understood through exponential functions of the form $\exp(\lambda t)$. If $\lambda < 0$, then $\exp(\lambda t)$ decays as t increases, and if $\lambda > 0$, then $\exp(\lambda t)$ grows as t increases.

3.1.1 Stable Node / Sink

Consider the diagonal system with state vector $\mathbf{x}(t) = (x_1(t), x_2(t))^T$:

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{x}(t) = \begin{pmatrix} c_1 e^{-2t} \\ c_2 e^{-t} \end{pmatrix}, \quad \lambda_1 = -2, \lambda_2 = -1.$$

Multiplying by A gives $x_1' = -2x_1$ and $x_2' = -x_2$. Each scalar equation has an exponential solution, giving the displayed vector with constants c_1 and c_2 fixed by the initial condition. Both components tend to zero, so the origin is an asymptotically stable node or sink.

The x_1 direction decays faster. Unless $c_2 = 0$, the slower term $\exp(-t)$ determines the approach to the origin, and trajectories become tangent to the x_2 axis. Initial conditions on either coordinate axis remain on that axis.

Any real 2×2 matrix with these distinct eigenvalues is similar to this diagonal matrix. An invertible change of coordinates can stretch, rotate, or shear its phase portrait, but preserves asymptotic stability. Section 3.1.4 makes the change in eigendirections explicit.

3.1.2 Unstable Node / Source

Changing the signs of the diagonal entries gives

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad x(t) = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^t \end{pmatrix}, \quad \lambda_1 = 2, \lambda_2 = 1.$$

Every nonzero solution grows without bound in norm, so the origin is an unstable node or source. A component remains zero when its coefficient is zero. For $c_1 \neq 0$, the faster term $\exp(2t)$ eventually dominates, aligning the trajectory with the x_1 direction far from the origin. The coordinate axes are invariant eigendirections.

3.1.3 Saddle Point

For a saddle, the two real eigenvalues have opposite signs:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x(t) = \begin{pmatrix} c_1 e^t \\ c_2 e^{-t} \end{pmatrix}, \quad \lambda_1 = 1, \lambda_2 = -1.$$

The line $x_1 = 0$ is the stable eigendirection: initial conditions on it converge to the origin. The line $x_2 = 0$ is the unstable eigendirection. Every solution with $c_1 \neq 0$ eventually leaves any bounded neighborhood of the origin. One expanding direction is therefore sufficient for instability, even though solutions on the stable line approach the equilibrium.

3.1.4 The Geometric Role of Eigenvectors

For distinct real eigenvalues, the eigenvalues determine the exponential rates and stability, while eigenvectors determine invariant lines and the geometric orientation of trajectories. Compare the following two matrices and the solution for the second:

$$\begin{aligned} A_1 &= \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, & A_2 &= \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}, \\ u_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & u_2 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & A_2 u_1 &= -2u_1, & A_2 u_2 &= -u_2, \\ x(t) &= c_1 e^{-2t} u_1 + c_2 e^{-t} u_2. \end{aligned}$$

Both matrices have eigenvalues -2 and -1 . For A_1 , the invariant eigendirections are the coordinate axes. For A_2 , they are $x_2 = 0$ and $x_2 = x_1$. If $c_2 \neq 0$, the term with $\exp(-t)$ decays more slowly and controls the direction of approach: trajectories become tangent to $x_2 = x_1$ near the origin. If $c_2 = 0$, the trajectory stays on $x_2 = 0$. This shows how identical eigenvalues can coexist with different geometric orientations.

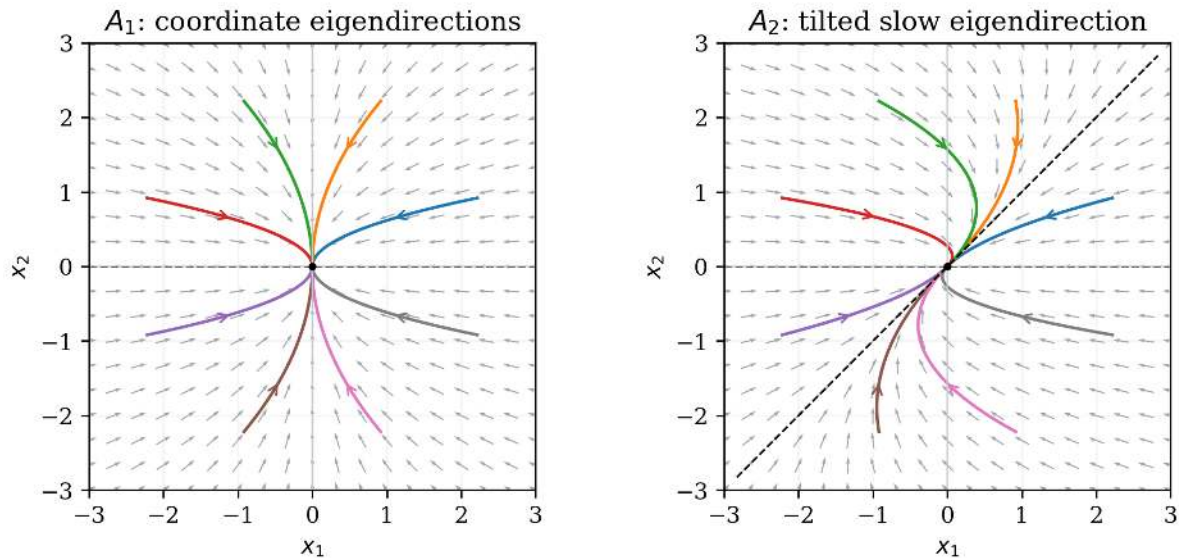


Figure 1. Two stable nodes with the same pair of distinct eigenvalues. Dashed lines mark eigendirections. The same initial conditions and axes make the effect of the eigenvectors visible.

3.2 Systems with Complex Eigenvalues

For a real 2×2 matrix, nonreal eigenvalues occur as a conjugate pair $a \pm bi$, with $b > 0$. Euler's formula and the corresponding real rotation-dilation block give

$$B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \lambda = a \pm bi, \quad b > 0,$$

$$e^{(a+bi)t} = e^{at}(\cos bt + i \sin bt),$$

$$e^{Bt} = e^{at} \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix}.$$

The factor $\exp(at)$ controls the exponential envelope and the trigonometric terms describe rotation. Thus $a < 0$ gives a spiral sink, $a = 0$ a center, and $a > 0$ a spiral source. In these block coordinates the angular speed is b and the oscillation period is $2\pi/b$.

For a general real matrix with the same nonreal pair, the real and imaginary parts of a complex eigenvector provide a real change of coordinates to such a block, up to the sign of rotation. The coordinate change can stretch and shear trajectories; eigenvalues alone do not specify clockwise versus counterclockwise rotation or constant Euclidean radius in the original coordinates.

3.2.1 Center

The center example is

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \det(\lambda I - A) = \lambda^2 + 1, \quad \lambda = \pm i.$$

Here $a = 0$ and $b = 1$. The solution is a rotation of the initial vector, so its Euclidean norm stays constant and every nonzero trajectory is a circle with period 2π . The origin is stable but not asymptotically stable. For a general real matrix with purely imaginary nonzero eigenvalues, an invertible coordinate change transforms these circles into ellipses.

3.2.2 Spiral Sink / Stable Spiral

The spiral-sink example is

$$A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad \det(\lambda I - A) = (\lambda + 1)^2 + 1, \quad \lambda = -1 \pm i.$$

Here $a = -1$ and $b = 1$. The exponential factor $\exp(-t)$ contracts the radius while the rotation continues. Every solution tends to the origin, which is asymptotically stable.

3.2.3 Spiral Source / Unstable Spiral

The spiral-source example is

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \det(\lambda I - A) = (\lambda - 1)^2 + 1, \quad \lambda = 1 \pm i.$$

Here $a = 1$ and $b = 1$. Every nonzero solution rotates while its norm grows as $\exp(t)$, so the origin is unstable. This reverses the attraction in the previous example while retaining the oscillation frequency.

3.3 Summary of Classification

The examples in this section show that eigenvalues allow us to classify the qualitative behavior of a system without explicitly computing every trajectory. For nonzero real eigenvalues, their signs determine whether the system is a stable node, unstable node, or saddle. For complex eigenvalues, the real part determines whether trajectories move inward, remain neutral, or move outward, while the imaginary part produces rotation.

Table 1 summarizes the relationship between eigenvalue type, phase-portrait geometry, and stability.

Table 1. Classification of Phase Portraits by Eigenvalue Type

Eigenvalue Type	System Behavior	Stability
$\lambda_1, \lambda_2 < 0; \lambda_1 \neq \lambda_2$	Stable node / sink	Asymptotically stable
$\lambda_1, \lambda_2 > 0; \lambda_1 \neq \lambda_2$	Unstable node / source	Unstable
$\lambda_1 > 0, \lambda_2 < 0$	Saddle point	Unstable
$\lambda = \pm bi; b > 0$	Center	Stable, not asymptotically stable
$\lambda = a \pm bi; a < 0, b > 0$	Spiral sink	Asymptotically stable
$\lambda = a \pm bi; a > 0, b > 0$	Spiral source	Unstable

Eigenvalue Type	System Behavior	Stability
$\lambda_1 = \lambda_2 = \lambda \neq 0; \Delta = 0$	Star node: eigenspace dimension 2. Improper node: dimension 1.	Asymptotically stable if $\lambda < 0$; unstable if $\lambda > 0$

In the complex rows, $b > 0$. For repeated nonzero eigenvalues, the eigenspace dimension distinguishes a star node from an improper node, as worked out in Section 4.1. Zero eigenvalues require separate treatment. If $D = 0$ and $T < 0$, the origin is stable but not asymptotically stable: solutions approach a point on a line of equilibria, not necessarily the origin. If $D = 0$ and $T > 0$, the origin is unstable. At $T = D = 0$, $A = 0$ makes every point an equilibrium, whereas a nonzero A has a nilpotent Jordan block and is unstable because some solutions grow linearly.

This classification uses eigenvalues and, in repeated cases, eigenspace dimensions. For 2×2 matrices, the eigenvalues can also be expressed through the trace and determinant. The next section uses this relationship to study how changes in matrix parameters affect the phase portrait.

4. The Effect of Changes in Matrix Parameters

After classifying the basic phase portraits, we can ask a more specific question: what happens when the entries of the matrix change? For two-dimensional systems, two especially important quantities are the trace and determinant of the matrix. For a 2×2 matrix, these quantities are enough to write the characteristic equation.

4.1 Trace and Determinant as Matrix Parameters

For an arbitrary real 2×2 matrix, define its trace T , determinant D , and discriminant Δ as follows:

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad T = p + s, \quad D = ps - qr,$$

$$\det(\lambda I - A) = (\lambda - p)(\lambda - s) - qr = \lambda^2 - T\lambda + D,$$

$$\lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4D}}{2}, \quad \Delta = T^2 - 4D.$$

The sign of Δ separates distinct real eigenvalues ($\Delta > 0$), repeated eigenvalues ($\Delta = 0$), and a nonreal conjugate pair ($\Delta < 0$). The boundary $\Delta = 0$ is the parabola $D = T^2/4$.

This curve separates the region of real eigenvalues from the region of complex eigenvalues on the trace-determinant plane. The trace-determinant plane is a standard tool for classifying two-dimensional linear systems (Strogatz, 2015; MIT OpenCourseWare, 2011b).

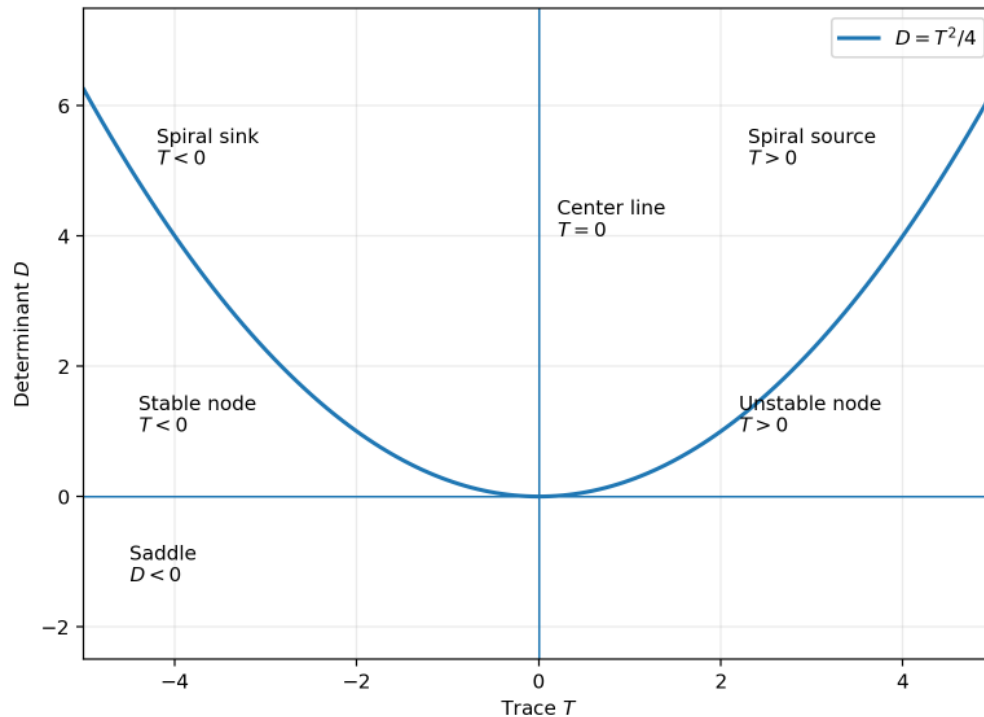


Figure 2. The trace-determinant plane for two-dimensional linear systems. The curve $D = T^2/4$ separates real and complex eigenvalues; the signs of T and D determine the main stability regions.

From the eigenvalue formula, we can see that the eigenvalues of a two-dimensional system depend only on the trace and determinant. Therefore, if the entries of the matrix change, instead of tracking all four entries separately, we can track the movement of the point (T, D) on the trace-determinant plane.

Although T and D determine the characteristic polynomial, they do not determine all eigenspace geometry. On $\Delta = 0$, a nonzero repeated eigenvalue may have a two-dimensional eigenspace (star node) or a one-dimensional eigenspace (improper node). Compare $-I$ and the Jordan block J :

$$\begin{aligned}
 -I &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = -I + N, \\
 N &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad N^2 = 0, \quad e^{Jt} = e^{-t}(I + tN), \\
 e^{Jt} &= e^{-t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad x(t) = e^{-t} \begin{pmatrix} c_1 + tc_2 \\ c_2 \end{pmatrix}, \\
 \frac{x_2(t)}{x_1(t)} &= \frac{c_2}{c_1 + tc_2} \rightarrow 0 \quad (c_2 \neq 0, t \rightarrow \infty).
 \end{aligned}$$

Both have $T = -2$, $D = 1$, and eigenvalue -1 . For $-I$, the solution is $\exp(-t)x(0)$, so each trajectory remains on its initial ray. For J , the nilpotent part satisfies $N^2 = 0$, so its exponential terminates after $I + tN$. This gives the displayed solution and the polynomial factor $t \exp(-t)$.

For J , the eigenvector $u = (1, 0)^T$ spans the line $x_2 = 0$. The vector $w = (0, 1)^T$ is a generalized eigenvector satisfying $(J + I)w = u$. If $c_2 = 0$, the trajectory stays on the eigenline. If $c_2 \neq 0$, the displayed ratio tends to zero, so the trajectory approaches the origin tangent to that line. Both systems are asymptotically stable, since $\exp(-t)$ and $t \exp(-t)$ tend to zero, but their trajectory geometry differs (Strang, 2016).

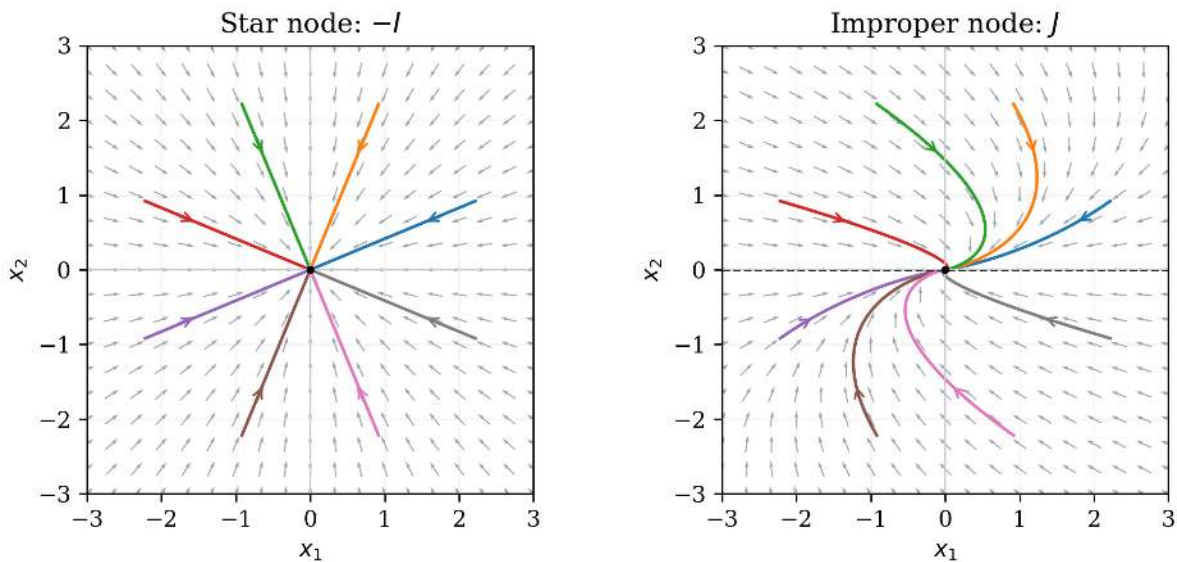


Figure 3. Equal eigenvalues with different eigenspace dimensions. The star node preserves every ray; the improper node has one eigenline, shown dashed. The same initial conditions and axes are used in both panels.

By Vieta's formulas, the determinant equals the product of the eigenvalues:

$$D = \lambda_1 \lambda_2.$$

If $D < 0$, then the eigenvalues have opposite signs, so the system has saddle behavior. If $D = 0$, then at least one eigenvalue is zero, which creates a boundary case. If $D > 0$, then the eigenvalues either have the same sign or form a complex conjugate pair.

By the same formula, the trace equals the sum of the eigenvalues:

$$T = \lambda_1 + \lambda_2.$$

In the complex eigenvalue case, the trace determines the real part:

$$\operatorname{Re}(\lambda) = T/2.$$

For a nonreal conjugate pair, $T < 0$ implies that all solutions tend to the origin; $T > 0$ implies that every nonzero solution grows without bound in norm as $t \rightarrow \infty$. At $T = 0$, nonzero solutions are periodic and bounded. These statements concern long-term behavior: distance from the origin need not vary monotonically in general coordinates.

For this reason, changes in the entries of A can be studied by tracking how T and D change.

4.2 One-Parameter Family

Consider a family in which the parameter a changes both trace and determinant:

$$A(a) = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}, \quad T(a) = 2a, \quad D(a) = a^2 + 1,$$
$$\det(\lambda I - A(a)) = (\lambda - a)^2 + 1, \quad \lambda = a \pm i,$$
$$D = \frac{T^2}{4} + 1.$$

The eigenvalues remain complex for every real a . Their real part is a and their imaginary parts are ± 1 . Consequently, $a < 0$ produces a spiral sink, $a = 0$ produces a center, and $a > 0$ produces a spiral source. In the coordinates used here, the radius is multiplied by $\exp(at)$, while the angular speed is 1.

Eliminating a gives $D = T^2/4 + 1$. This curve lies strictly above the repeated-eigenvalue boundary $D = T^2/4$, so it stays in the complex region. Crossing $T = 0$ changes the sign of the real part without changing the oscillation frequency. Figure 4 shows this path, and Table 2 summarizes the three cases.

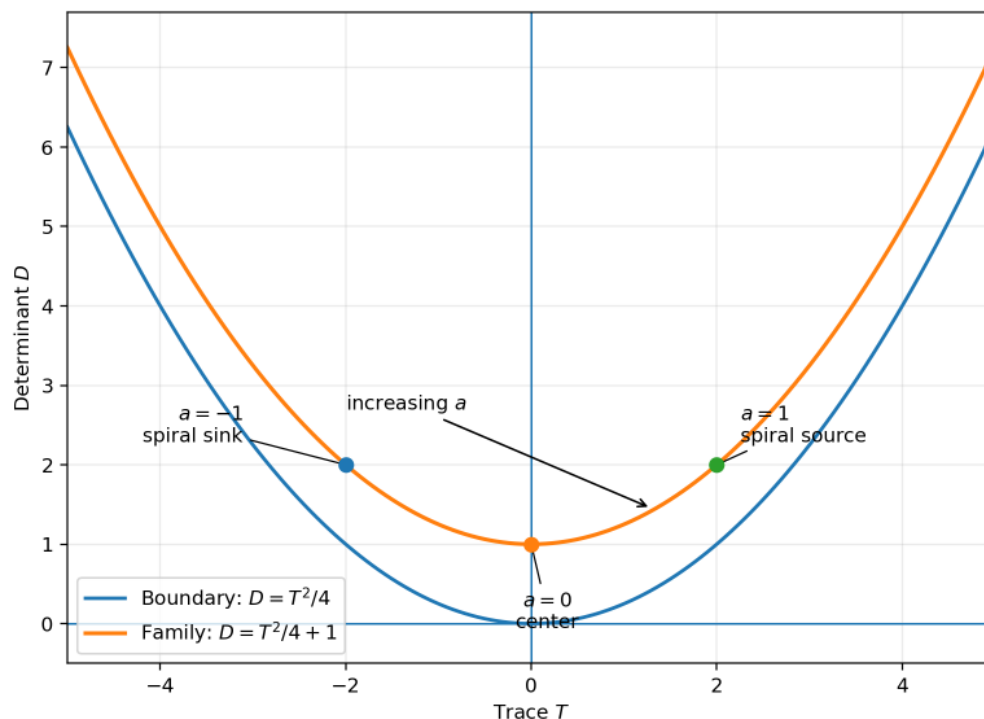


Figure 4. The one-parameter family $A(a)$ on the trace-determinant plane. As a increases, the point follows $D = T^2/4 + 1$, crosses $T = 0$, and changes from a spiral sink to a center and then to a spiral source.

Table 2. Behavior of the Family $A(a)$ for Different Signs of a

Parameter	Eigenvalues	Trace	Behavior
$a < 0$	$\lambda = a \pm i$	$T < 0$	spiral sink
$a = 0$	$\lambda = \pm i$	$T = 0$	center
$a > 0$	$\lambda = a \pm i$	$T > 0$	spiral source

In this example, changing a moves the system along one curve in the trace-determinant plane.

4.3 Fixed Determinant and Changing Trace

Keeping D fixed makes it easier to see what changing T does by itself. In previous examples, the trace and determinant could change at the same time. Here, the determinant remains constant, so changes in the phase portrait can be connected directly to changes in the trace.

Fix $D = 2$ and compare $T = -2, 0$, and 2 using the real matrix family

$$A_T = \begin{pmatrix} T/2 & -\omega(T) \\ \omega(T) & T/2 \end{pmatrix}, \quad \omega(T) = \sqrt{2 - \frac{T^2}{4}},$$

$$\text{tr}(A_T) = T, \quad \det(A_T) = 2, \quad |T| \leq 2\sqrt{2},$$

$$\lambda = \frac{T}{2} \pm i\sqrt{2 - \frac{T^2}{4}} \quad (|T| < 2\sqrt{2}).$$

The family is real for $|T| \leq 2\sqrt{2}$; the three chosen values lie strictly inside the complex-eigenvalue region. Its solutions are rotations multiplied by $\exp(Tt/2)$, with frequency $\omega(T)$. Therefore, the sign of T changes the long-term attraction or repulsion, while $\omega(T)$ also varies.

Case 1: $T = -2$

At $T = -2$, the eigenvalues are $\lambda = -1 \pm i$.

The real part is -1 , which is negative. Therefore, the exponential factor causes decay, and the trajectories spiral toward the origin. This is a spiral sink.

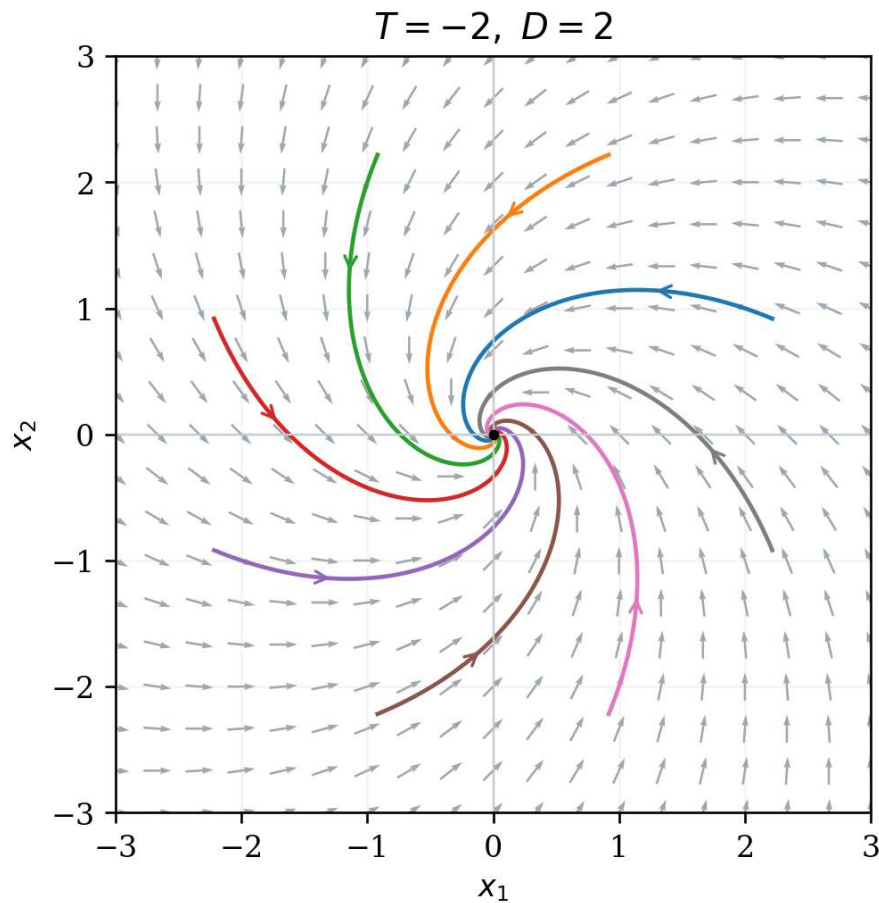


Figure 5a. Fixed $D = 2$, $T = -2$: eigenvalues $-1 \pm i$; spiral sink.

Case 2: $T = 0$

At $T = 0$, the eigenvalues are $\lambda = \pm i\sqrt{2}$.

The real part is 0. Therefore, there is no exponential growth or decay. Trajectories rotate around the origin without moving inward or outward. This is a center.

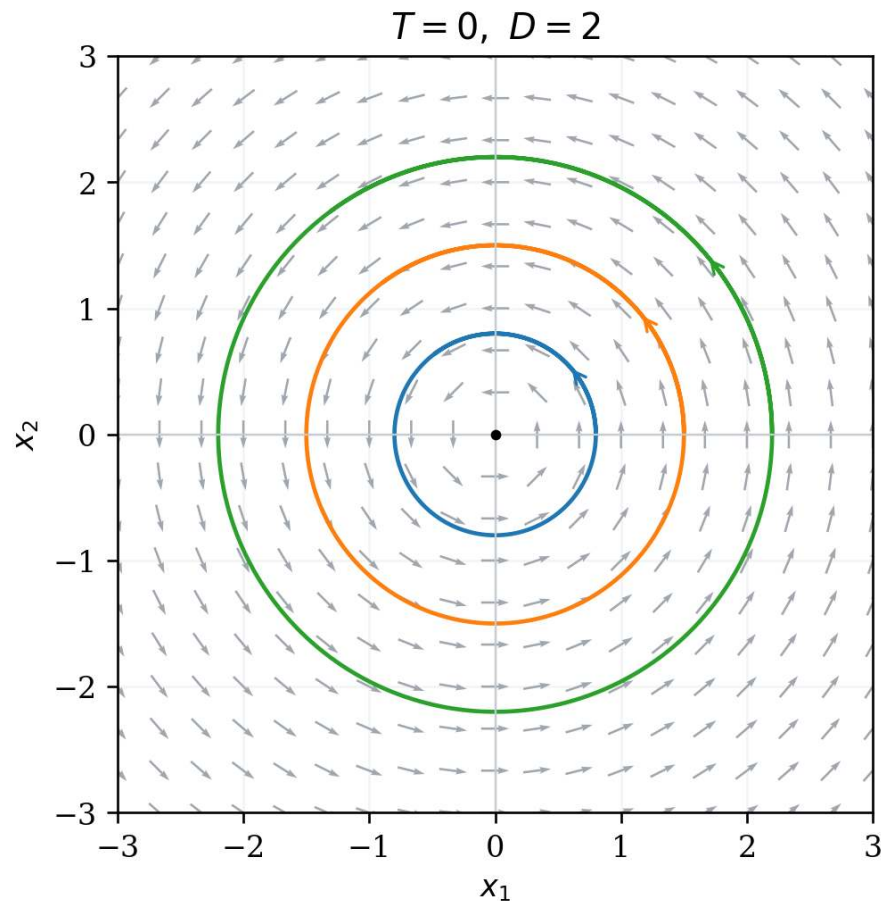


Figure 5b. Fixed $D = 2$, $T = 0$: eigenvalues $\pm i\sqrt{2}$; center.

Case 3: $T = 2$

At $T = 2$, the eigenvalues are $\lambda = 1 \pm i$.

The real part is 1, which is positive. Therefore, the exponential factor causes growth, and the trajectories spiral away from the origin. This is a spiral source.

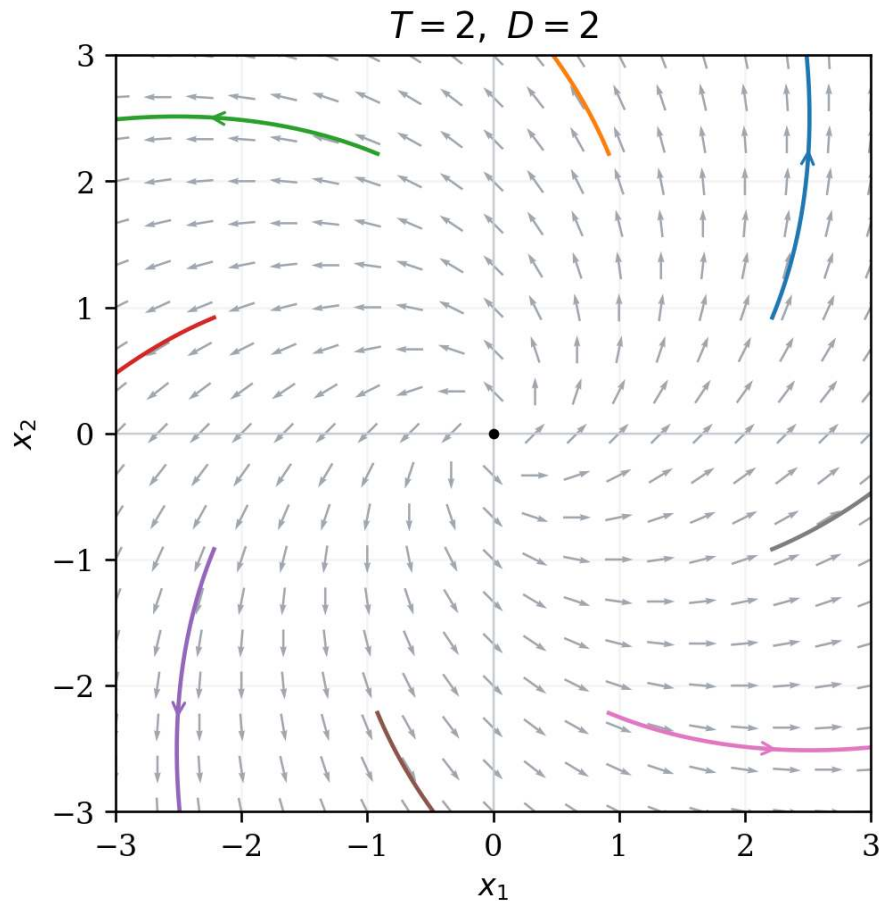


Figure 5c. Fixed $D = 2$, $T = 2$: eigenvalues $1 \pm i$; spiral source.

Role of the Trace

The determinant remains 2 while T takes the values -2 , 0 , and 2 . The common real part is respectively -1 , 0 , and 1 , producing a spiral sink, a center, and a spiral source. The oscillation frequency also changes: it is 1 , $\sqrt{2}$, and 1 in the three cases. Thus, this comparison isolates the effect of changing trace at fixed determinant, but does not hold frequency constant.

Two Explanations for the Role of the Trace

The algebraic explanation follows from $\text{Re}(\lambda) = T/2$ in the complex region. The geometric explanation concerns the area transported by the flow. For A with entries p, q, r, s , the divergence and area evolution are

$$\begin{aligned} \text{div}(Ax) &= \frac{\partial(px_1 + qx_2)}{\partial x_1} + \frac{\partial(rx_1 + sx_2)}{\partial x_2} = p + s = T, \\ \det(e^{At}) &= e^{t\text{tr}(A)} = e^{Tt}, \quad \mathcal{A}(t) = e^{Tt}\mathcal{A}(0). \end{aligned}$$

Here, $\mathcal{A}(t)$ denotes the area of a region after time t . A negative trace contracts area, a positive trace expands it, and zero trace preserves it. This does not imply that all trajectories contract or

expand in norm. A saddle may contract area while expanding along one eigendirection; the example $T = -2$, $D = -1$ in Section 4.4 demonstrates this distinction.

4.4 Fixed Trace and Changing Determinant

Keeping T fixed allows the effect of D to be followed along a vertical line in the trace-determinant plane. The determinant changes the discriminant and may also change stability. Only in the complex region is the common real part fixed by the trace.

Use the following continuous family, defined for every real D :

$$A_D = \begin{pmatrix} -1 & -1 \\ D-1 & -1 \end{pmatrix}, \quad \text{tr}(A_D) = -2, \quad \det(A_D) = D, \quad \lambda = -1 \pm \sqrt{1-D}.$$

Substitution into the quadratic formula gives the displayed eigenvalues. Changing D alters the square-root term and can change the signs of the real eigenvalues. If $D < 0$, the product of the eigenvalues is negative. Then one eigenvalue is positive and the other is negative, resulting in saddle behavior. For example, setting $D = -1$

$$\lambda = -1 \pm \sqrt{1 - (-1)} = -1 \pm \sqrt{2}.$$

Since one of these values is positive and the other is negative, the phase portrait is a saddle. So a negative trace is not enough to guarantee stability when $D < 0$.

At $D = 0$, the eigenvalues are 0 and -2 .

The zero eigenspace is the line $x_2 = -x_1$, and every point on it is an equilibrium. The other eigendirection, $x_2 = x_1$, decays at rate 2. Solutions therefore converge to a point on the equilibrium line determined by the initial condition. The origin is stable but not asymptotically stable, since nearby nonzero equilibria do not converge to it.

If $0 < D < 1$, both eigenvalues are real and negative. For instance, with $D = 0.5$

$$\lambda = -1 \pm \sqrt{1 - 0.5} = -1 \pm \sqrt{0.5}.$$

Both eigenvalues are negative and distinct, so this is a stable node. Unless the coefficient of the slower-decaying mode is zero, trajectories approach the origin tangent to its eigendirection. Initial conditions on the faster eigendirection remain on that line.

At $D = 1$, the repeated eigenvalue is $\lambda = -1$.

For this family, $D = 1$ gives a stable improper node, not a star node: there is only one independent eigenvector, along $x_2 = 0$. The sign of the off-diagonal entry is opposite to J in Section 4.1, so the corresponding solution is

$$A_1 = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \quad x(t) = e^{-t} \begin{pmatrix} c_1 - tc_2 \\ c_2 \end{pmatrix}.$$

For nonzero c_2 , the trajectories approach the origin tangent to $x_2 = 0$. The polynomial factor does not prevent convergence because $t \exp(-t)$ tends to zero.

If $D > 1$, the expression $1 - D$ becomes negative, causing the eigenvalues to become complex. For example, when $D = 2$

$$\lambda = -1 \pm i.$$

The real part is -1, so trajectories converge toward the origin. The imaginary part introduces rotation, transforming the phase portrait into a spiral sink.

If the determinant is increased further, the real part remains the same, but the imaginary part increases. For instance, at $D = 5$

$$\lambda = -1 \pm 2i.$$

This is also a spiral sink. Its oscillation period is π , compared with 2π at $D = 2$. These are periods of the oscillatory factor; the instantaneous angular speed in the original coordinates need not be constant.

On the trace-determinant plane, fixing $T = -2$ means that the system moves along the vertical line $T = -2$. As D increases, the point moves upward along this line. This movement crosses several regions: saddle, boundary case, stable node, repeated eigenvalue, and spiral sink.

Figure 6 displays representative members of this family, including both boundary cases.

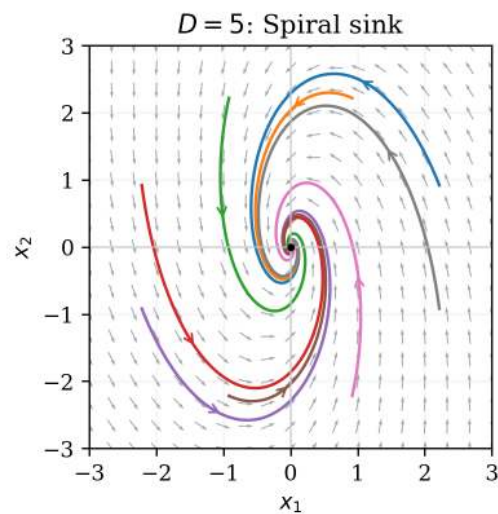
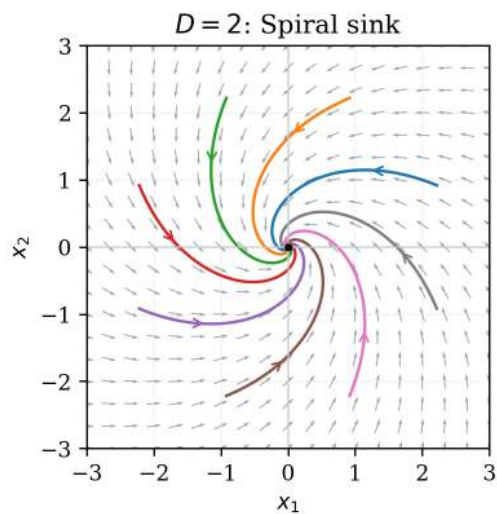
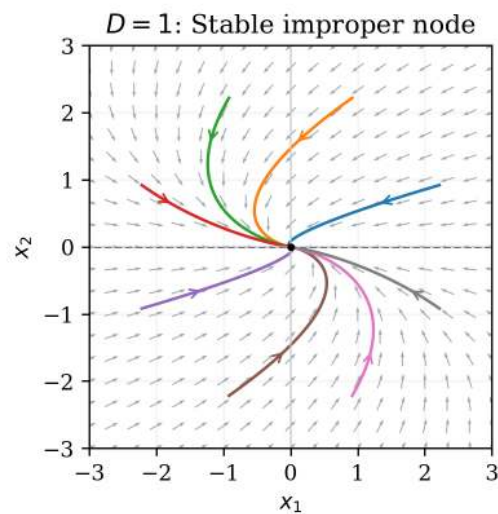
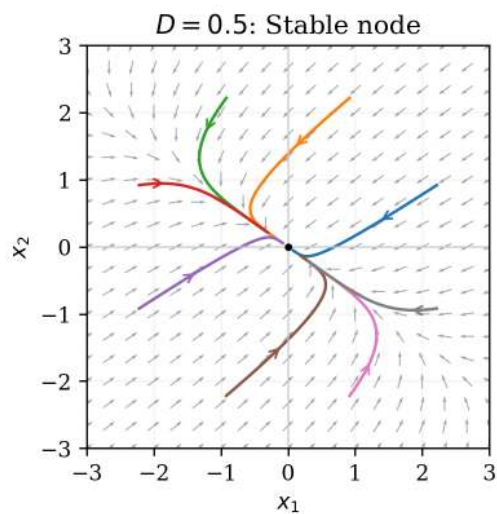
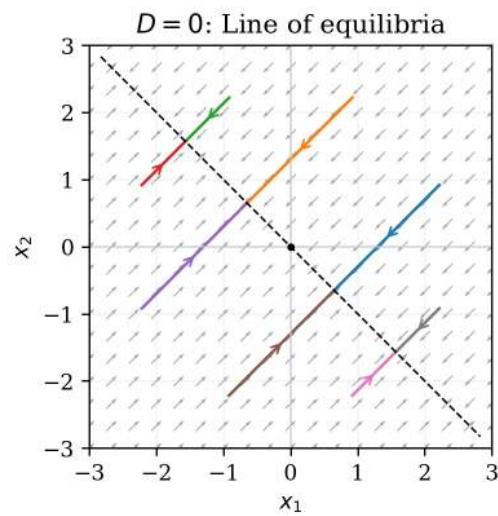
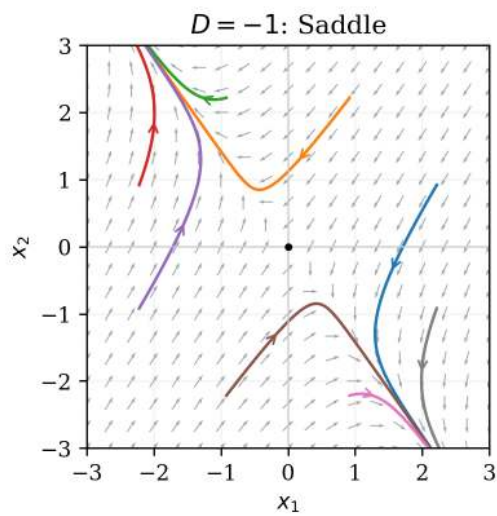


Figure 6. Phase portraits for the fixed-trace family A_D . Panels show $D = -1, 0, 0.5, 1, 2$, and 5 . The $D = 0$ panel marks a line of equilibria; the $D = 1$ panel is a stable improper node. Arrows indicate forward-time direction.

At $T = -2$, the discriminant is $\Delta = T^2 - 4D = 4 - 4D$.

As D increases, $\Delta = 4 - 4D$ decreases. The family passes through a saddle ($D < 0$), a line of equilibria ($D = 0$), a stable node ($0 < D < 1$), a stable improper node ($D = 1$), and a spiral sink ($D > 1$). The passage through $D = 0$ changes asymptotic stability, whereas the passage through $D = 1$ changes the phase-portrait geometry within the stable region.

More generally, at fixed T , changing D changes the discriminant $\Delta = T^2 - 4D$ and can change stability when D crosses zero. For $T = -2$, the saddle at $D < 0$ becomes asymptotically stable when $D > 0$. Within the complex region $D > 1$, both eigenvalues have real part -1 , while the oscillation frequency is $\sqrt{D - 1}$. Thus, independence of the attraction rate from D applies within the complex region, not across the whole family.

4.5 Discussion of Long-Term Behavior

The trace-determinant plane gives a compact decision procedure for a real 2×2 system. Its characteristic equation is

$$\lambda^2 - T\lambda + D = 0,$$

with discriminant $\Delta = T^2 - 4D$.

If $D < 0$, the eigenvalues have opposite signs and the equilibrium is a saddle. If $D > 0$ and $T < 0$, the origin is asymptotically stable; if $D > 0$ and $T > 0$, it is unstable. Within either half-plane, $\Delta > 0$ gives a node, $\Delta = 0$ gives a repeated-eigenvalue boundary case, and $\Delta < 0$ gives a spiral. When $T = 0$ and $D > 0$, the eigenvalues are purely imaginary and the linear system is a center.

The two parameter studies separate mechanisms under explicit conditions. In the complex region, changing T at fixed D changes the real part $T/2$ and can turn a spiral sink into a center and then a spiral source. At fixed T , changing D can turn a saddle into an asymptotically stable node; only after entering the complex region does the real part remain constant while the oscillation frequency varies. Trace measures the exponential rate of area change, whereas stability requires the combined spectral information encoded by T and D . At repeated or zero eigenvalues, the eigenspace and Jordan structure complete the classification.

4.6 Engineering Application: Damped Mechanical Oscillator

A mass–spring–damper provides a physical interpretation of the preceding classification. Let $y(t)$ denote displacement from equilibrium, $m > 0$ the mass, $k > 0$ the spring stiffness, and $c \geq 0$ the viscous damping coefficient. Assuming small displacements, constant parameters, and no external forcing, Newton's second law gives

$$m y'' + c y' + k y = 0.$$

Set $x_1 = y$ and $x_2 = y'$. The equivalent first-order system is

$$x_1' = x_2, \quad x_2' = -(k/m)x_1 - (c/m)x_2.$$

Thus A has first row $(0, 1)$ and second row $(-k/m, -c/m)$. Its spectral quantities are

$$T = -c/m, \quad D = k/m, \quad \lambda_{1,2} = [-c \pm \sqrt{c^2 - 4mk}]/(2m).$$

This standard mechanical model connects second-order differential equations to the phase-plane methods used throughout this paper (Boyce et al., 2017; Strogatz, 2015). Holding m and k fixed while varying c moves horizontally in the trace–determinant plane at fixed $D > 0$, giving a physical counterpart to Section 4.3.

For $0 < c < 2\sqrt{mk}$, the eigenvalues form a complex conjugate pair with negative real part. The origin is a spiral sink: displacement oscillates with an envelope proportional to $\exp[-ct/(2m)]$ and damped angular frequency $\sqrt{k/m - c^2/(4m^2)}$. At $c = 0$, the eigenvalues are purely imaginary and the ideal unforced system is a center with no energy loss.

At $c = 2\sqrt{mk}$, the system is critically damped. The repeated eigenvalue is $-\sqrt{k/m}$, and A has only one independent eigenvector. The solution has the form $y(t) = (C_1 + C_2 t)\exp[-\sqrt{k/m}t]$, linking critical damping directly to the Jordan-block analysis in Section 4.1.

For $c > 2\sqrt{mk}$, both eigenvalues are distinct and negative. The origin is a stable node and the response is a sum of two decaying exponentials, without sustained oscillation. Increasing c indefinitely does not guarantee faster settling: the slower eigenvalue approaches $-k/c$ as c becomes large. Critical damping is therefore a boundary between oscillatory and nonoscillatory modes, not a universal optimum for every initial condition or design objective.

For a concrete comparison, take $m = 1$ kg and $k = 4$ N/m. With $c = 2$ N·s/m, $\lambda = -1 \pm i\sqrt{3}$ s⁻¹ (underdamped); with $c = 4$ N·s/m, $\lambda = -2$ s⁻¹ twice (critical); and with $c = 6$ N·s/m, $\lambda = -3 \pm \sqrt{5}$ s⁻¹ (overdamped). For the common initial conditions $y(0) = y_0$ and $y'(0) = 0$, the critical response is $y(t) = y_0(1 + 2t)\exp(-2t)$, with t measured in seconds. These are illustrative model parameters, not measurements from a particular device.

The energy $E = \frac{1}{2}m(y')^2 + \frac{1}{2}ky^2$ satisfies $E' = -c(y')^2$. Positive damping removes energy, consistent with asymptotic stability. A formal $c < 0$ represents energy input rather than a passive damper and yields positive-real-part eigenvalues and instability.

This model is a building block for vibration and suspension analysis, but an actual vehicle or structure generally requires forcing and additional degrees of freedom. The unforced series RLC equation $Lq'' + Rq' + q/C = 0$ has the same mathematical form under $m \leftrightarrow L$, $c \leftrightarrow R$, and $k \leftrightarrow 1/C$. The application illustrates how eigenvalues inform engineering interpretation without claiming a tested design or a new physical result.

5. Conclusion

The goal of this paper was to explain how the eigenvalues and eigenvectors of A affect solutions of $x'(t) = Ax(t)$. Eigenvalues determine the exponential growth and decay rates. Eigenspaces determine invariant directions, while Jordan structure supplies additional polynomial factors in non-diagonalizable cases. Complex conjugate eigenvalues describe oscillatory motion whose exponential envelope is controlled by their real part.

Diagonalization explains why these quantities appear in the solution. When A has enough independent eigenvectors, a change to eigenvector coordinates separates the system into scalar exponential equations. The eigenvalues then determine the rates of growth and decay, while the eigenvectors determine the invariant directions and the orientation of the phase

portrait. Repeated-eigenvalue examples show why the eigenvector structure cannot be ignored even when the trace and determinant are known.

For real 2×2 systems, trace and determinant determine the characteristic polynomial. In the complex region, fixing the determinant and varying the trace changes both the common real part and the oscillation frequency; the sign of the trace determines attraction or repulsion. At fixed trace, varying the determinant can change stability when a zero eigenvalue is crossed. Once the eigenvalues are complex, their real part remains fixed while their imaginary part changes.

The classification used here is classical. The contribution of this expository study is the explicit comparison of matrices with the same eigenvalues, the solution of a repeated-eigenvalue example, and the construction of parameter families that connect algebraic conditions to phase portraits. These comparisons distinguish spectral information from eigenspace geometry and show why trace alone is not a stability criterion.

The scope is limited to real two-dimensional autonomous linear systems. The repeated-eigenvalue example includes a non-diagonalizable matrix, but higher-dimensional Jordan blocks and interactions between several invariant subspaces are not treated. Nonlinear systems and the limits of local linearization provide further directions for study.

5.1 Open Questions and Future Directions

The analysis raises several natural extensions. First, eigenvalues determine asymptotic behavior for autonomous linear systems, but nonnormality can create substantial transient growth before decay. Quantifying that behavior requires singular values, pseudospectra, or bounds on $\exp(At)$, not eigenvalues alone (Higham, 2008; Trefethen & Embree, 2005). A computational extension could compare matrices with identical eigenvalues but different degrees of nonnormality and measure the maximum transient amplification. Recent work also emphasizes the distribution of likely transient amplification rather than only the worst-case optimal disturbance (Frame & Towne, 2024).

Second, in dimensions greater than two, several invariant subspaces and larger Jordan blocks can interact, so trace and determinant no longer provide a complete low-dimensional classification. Studying how Jordan structure, conditioning of the eigenvector matrix, and spectral gaps affect trajectories would generalize the comparisons developed here (Perko, 2001; Teschl, 2012).

Third, for nonlinear systems $x' = f(x)$, the Jacobian at an equilibrium connects this analysis to local linearization. Hyperbolic equilibria inherit their local qualitative behavior from the linearized system, whereas zero or purely imaginary eigenvalues require higher-order terms, center-manifold methods, or bifurcation analysis (Guckenheimer & Holmes, 1983; Perko, 2001). These cases mark the boundary of what the eigenvalues of one constant matrix can determine. Koopman operator theory offers a complementary modern direction by representing nonlinear dynamics through linear operators on observables and connecting spectral analysis with data-driven modeling (Brunton et al., 2022).

References

- [1] Boyce, W. E., DiPrima, R. C., & Meade, D. B. (2017). *Elementary differential equations and boundary value problems* (11th ed.). Wiley.
<https://books.google.com/books?id=SyaVDwAAQBAJ>
- [2] Brunton, S. L., Budišić, M., Kaiser, E., & Kutz, J. N. (2022). Modern Koopman theory for dynamical systems. *SIAM Review*, 64(2), 229-340. <https://doi.org/10.1137/21M1401243>
- [3] Frame, P., & Towne, A. (2024). Beyond optimal disturbances: A statistical framework for transient growth. *Journal of Fluid Mechanics*, 983, A2. <https://doi.org/10.1017/jfm.2024.100>
- [4] Guckenheimer, J., & Holmes, P. (1983). *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*. Springer. <https://doi.org/10.1007/978-1-4612-1140-2>
- [5] Higham, N. J. (2008). *Functions of matrices: Theory and computation*. Society for Industrial and Applied Mathematics. <https://doi.org/10.1137/1.9780898717778>
- [6] Hirsch, M. W., Smale, S., & Devaney, R. L. (2012). *Differential equations, dynamical systems, and an introduction to chaos* (3rd ed.). Academic Press.
<https://www.sciencedirect.com/book/9780123820105/differential-equations-dynamical-systems-and-an-introduction-to-chaos>
- [7] MIT OpenCourseWare. (2011a). *Qualitative behavior: Phase portraits*. In 18.03SC Differential Equations (Fall 2011). Massachusetts Institute of Technology.
<https://ocw.mit.edu/courses/18-03sc-differential-equations-fall-2011/pages/unit-iv-first-order-systems/qualitative-behavior-phase-portraits/>
- [8] MIT OpenCourseWare. (2011b). *Trace-determinant diagram* [Course notes]. In 18.03SC Differential Equations (Fall 2011). Massachusetts Institute of Technology.
https://ocw.mit.edu/courses/18-03sc-differential-equations-fall-2011/resources/mit18_03scf11_s34_5text/
- [9] Perko, L. (2001). *Differential equations and dynamical systems* (3rd ed.). Springer.
<https://doi.org/10.1007/978-1-4613-0003-8>
- [10] Strang, G. (2010). *Lecture 23: Differential equations and $\exp(At)$* [Video lecture]. In 18.06 Linear Algebra (Spring 2010). MIT OpenCourseWare, Massachusetts Institute of Technology.
<https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/resources/lecture-23-differential-equations-and-exp-at/>
- [11] Strang, G. (2016). *Introduction to linear algebra* (5th ed.). Wellesley-Cambridge Press.
<https://math.mit.edu/~gs/linearalgebra/>
- [12] Strogatz, S. H. (2015). *Nonlinear dynamics and chaos: With applications to physics, biology, chemistry, and engineering* (2nd ed.). Westview Press.
<https://www.routledge.com/Nonlinear-Dynamics-and-Chaos-With-Applications-to-Physics-Biology-Chemistry-and-Engineering/Strogatz/p/book/9780813349107>
- [13] Tarnowski, W., Neri, I., & Vivo, P. (2020). Universal transient behavior in large dynamical systems on networks. *Physical Review Research*, 2, 023333.
<https://doi.org/10.1103/PhysRevResearch.2.023333>
- [14] Teschl, G. (2012). *Ordinary differential equations and dynamical systems*. American Mathematical Society. <https://doi.org/10.1090/gsm/140>
- [15] Trefethen, L. N., & Embree, M. (2005). *Spectra and pseudospectra: The behavior of nonnormal matrices and operators*. Princeton University Press. <https://doi.org/10.2307/j.ctvzxx9kj>