“A BRIEF HISTORY OF IMAGINARY NUMBERS”

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ABSTRACT: This article focuses on the history of imaginary numbers and how they are used. I will explain the theory behind them, the questions that were answered by their invention, and one of the most known examples of how they are used.

CHAPTER 1:

1.1: REAL NUMBERS

To understand what imaginary numbers are, the reader must first need to understand what real numbers are. A real number is a value that represents a quantity along a continuous number line (1). Numbers like 1, 2, 3, 0, π, -3, and √2/2 are all real numbers. Real numbers are symbolized in math with the letter R. Real numbers are essentially the sum of some sets of numbers, including:

Natural Numbers: [1, 2, 3...]

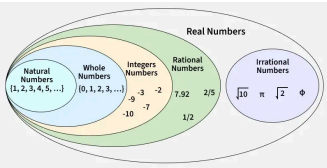
Whole Numbers: [0,1,2,3...]

Integers: [-2, -1, 0, 1, ...]

Rational Numbers: [2.1, 6.8 ,]

Irrational Numbers: [π, e, φ, ...]

If we combine all of this together, we get real numbers. As said before, real numbers can be visualized in a line called the real number line. Arithmetic operations can be done with them, such as addition, subtraction, multiplication, and division.



(Image 1: Real numbers)

1.2: INTRODUCTION TO IMAGINARY NUMBERS

Now that the reader is aware what real numbers are let's take a look at imaginary ones. An imaginary number is the square root of a negative number such as . This alone does not make sense, because how can we have the square root of a negative number?. The square root is the reverse of squaring a number, but which number, if we multiply it by itself, will give us a negative number? To answer these questions, imaginary numbers were introduced in mathematics.

CHAPTER 2

2.1: BRIEF HISTORY OF IMAGINARY NUMBERS

Imaginary numbers have an interesting history on how they have been shaped, how they are solved, and how they have contributed to form complex numbers by the help of Greek mathematician, Heron of Alexandria, who lived sometime between 100 BC and 100 AD. They first appeared in a study concerned with the dimensions of a pyramidal frustum (1). Del Ferro, an Italian mathematician, was trying to find a formula for equations with the highest power of 3, or cubic ($ax^3 + bx^2 + cx + d = 0$). He thought that
this was very complicated, so instead he tried to solve the form ($ax^3 + cx = d$). He found a formula to solve this form of cubics, but there was a problem: for certain values of c and d, the formula would give negative numbers under a square root. Bombelli, an Italian mathematician, thought that if known numbers did not work, then there must be another type of numbers that would work. He defined that is a number, and because he couldn't find a place for it in the real plane, he extended the number line, such as many had done in the past for solving other types of problems. A very famous mathematician Euler was the first to use the letter i to represent the imaginary unit or . By researching it more mathematicians reached the conclusion that

$i = -1$

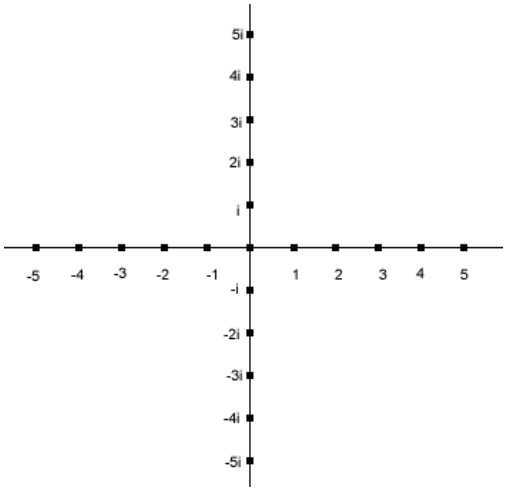
$i^2 = -1$

$i^3 = -i$

$i^4 = 1$.

If we keep increasing the exponents, we see that this pattern repeats every 4 digits such as in a cycle. For example, $i^6 = -1$, which is the same as i^2. We wanted to find a number that when multiplied by itself, it leads us to a negative number. For us to shape an imaginary unit from a $\sqrt{-n}$, we split the square root into two, $\sqrt{-1}$ and $\sqrt{n}$ and we compute it. An the example from the formula given above, $\sqrt{-16} = \sqrt{16} * \sqrt{-1} = 4i$. If we multiply two real numbers together,

two positive or negative ones, we get a positive outcome. But if we multiply a positive and a negative, we get a negative number. If we represent this in the number line, we see that we are doing a 180 degree turn. Therefore, what we need is a number that, when multiplied by itself, would turn 90°, not 180°. This is what imaginary numbers do. Therefore,from this,we can create the complex number line, where imaginary number is at a right angle to our number line (1). So the real number line could be represented as the x axis. Imaginary numbers introduced a new plane in the number line, which took the y-axis.



(Image 2)

CHAPTER 3

3.1 DEFINITION OF COMPLEX NUMBERS

As we saw, imaginary numbers introduced a new plane. When we then put together a real and an imaginary number, we get what we call a complex number (3+2i) (1). So complex numbers can be broken down to the real and the imaginary part (a+bi), where a is the real and b is the imaginary part, provided that a and b are real numbers. Imaginary numbers are a subset of complex numbers in the same form, a+bi, where a=0. Because we have extended the number line into two dimensions, 'a' represents the x-axis and 'bi' the y-axis. So every complex number can be expressed as Cartesian coordinates (x, y). Using the example 3+2i, 3 is x and 2i is y, so it can be written as (3, 2).

3.2: OPERATIONS OF COMPLEX

For real numbers we know how to do the 4 basic operations (addition, subtraction, multiplication, and division). How about the complex numbers?

3.2.1: ADDITION AND SUBTRACTION

Addition and subtraction of complex numbers are defined like vectors, adding or subtracting real and imaginary parts correspondingly (2). If we have two complex numbers, by the form of (a+ib) and (c+id), we split them into two, the real parts (a±c) and the imaginary parts i(b±d), and we compute them. For example:
(2+5i)+(3-4i)=(2+3)+i(5-4)=5+i
(2+5i)-(3-4i)=(2-3)+i(5+4)=-1+9i.

3.2.2: MULTIPLICATION

For multiplication, we can define it so we can use the commutative and distributive properties. (a+ib)(c+id)=a(c+di)+(ib)(c+di) (2). By extending it, we reach (ac-bd)+i(ad+bc), which is the formula for multiplying complex numbers.
Example: (2+3i)(3-4i)=(6+12)+i(-8+9)=18+i.

3.2.3: DIVISION

For division we use the so called complex conjugate. We keep the real part as it was and change the sign of the imaginary part. So if we have a number z where z=a+bi, Its conjugate number ($\bar{z}$) is $\bar{z}$=a-bi(3). Divisions are in the form $\frac{a+ib}{c+id}$. We find the denominator's conjugate and multiply both numerator and denominator with it: $\frac{a+ib}{c+id} = \frac{(a+ib)\cdot(c-id)}{(c+id)\cdot(c-id)}$. We can see that the denominator is now a difference of squares. Using that we get $\frac{(a+ib)*(c-id)}{c^2-(id)^2}$, and we continue using the formula for multiplication from 3.2.2.

Example: $\frac{1+i}{1-i} = \frac{(1+i)^2}{1^2-i^2} = \frac{2i}{2} = i$

CHAPTER 4

4.1: INTRODUCTION TO EULER'S EQUATION

Apart from other uses, the imaginary unit appears in Euler's most famous equation: $e^{i\pi} + 1 = 0$. This equation unifies analysis with e, the imaginary with i, geometry with π, the unit 1 and the representation of nothing, 0. Euler first discovered and published in his 1748 work "Introduction to analysis of the infinite", the formula $e^{ix} = cos(x) + i \cdot sin(x)$,
where setting x=π results in the equation.

4.2 FORMULAS FOR PROOF OF EULER'S EQUATION

To understand how this identity is made, we first need to see the base of it. The base of the equation is the exponential function e^x, which Euler published in the same work. Using a tool called the Maclaurin series, which can express a polynomial or a function as an infinite sum starting from 0 on the graph we can express e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots$$

(Image 3)

The same we can do for the cosine and sine:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

(Image 4)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

(Image 5)

4.3 SHORT PROOF OF EULER'S EQUATION

Now if we substitute ix=x, we get , which can be expanded using the same series.

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \cdots$$

(Image 6)

Splitting the infinite sum into two parts, we get the Maclaurin series for cos(x) + the same series for sin(x) multiplied by i. So we reach the conclusion that $e^{ix} = cos(x) + i \cdot sin(x)$ If we set x=π, we get $e^{i\pi} = -1 + 0$. Adding +1 into both parts, we get $e^{i\pi} + 1 = 0$. This is a brief proof of Euler's identity. Without i, we would not have e^ix and therefore no identity. This identity is regarded as one of the most beautiful equations ever written.

CONCLUSION

Imaginary numbers opened the door for problems deemed impossible at the time. Mathematicians faced difficulties solving problems that included square roots of negative numbers but found a way to bypass those problems by introducing a new type of number unknown at the time: the imaginary. Through the struggles of a couple of mathematicians to being part of the beauty of mathematics, imaginary numbers are essential for math to continue its journey: to answer questions about the world, to challenge some nerves, but overall to describe the world we live in. Also the analysis of imaginary numbers has it's own branch of mathematical analysis called Complex Analysis.

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